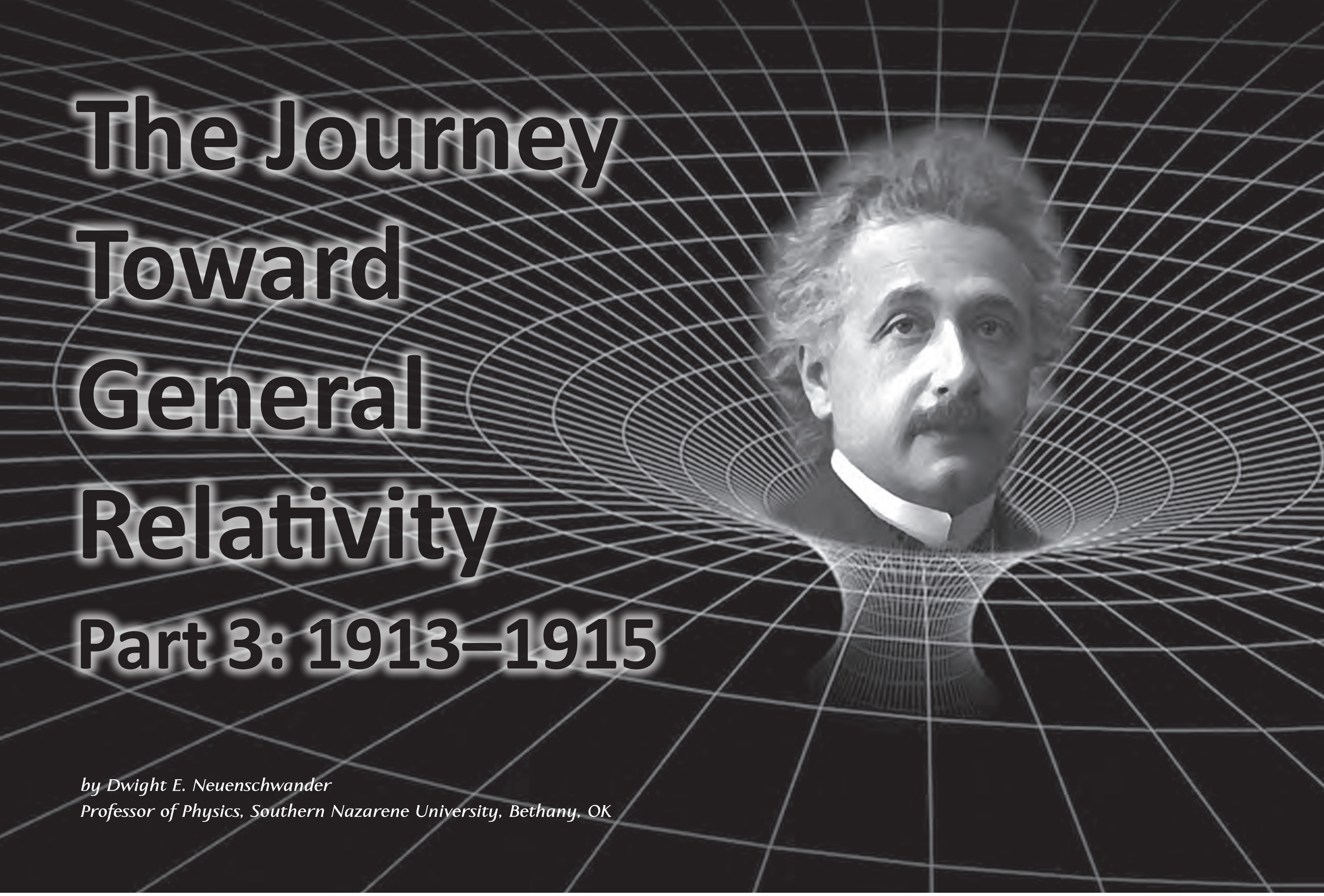

# The Journey Toward General Relativity

## Part 3: 1913–1915

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Image by Matt Payne

***This is the final installment in a three-part series which outlines Albert Einstein's development of the general theory of relativity.***

Part 1 recalled Einstein's "happiest thought," now called the principle of equivalence for gravitational and inertial mass, which started him down the path toward general relativity. Part 2 [1] focused on Einstein's application of Gaussian curved space to relativity. We saw how Einstein's task was to construct the field equations that determine the metric tensor components $g_{\mu\nu}$ by extending Poisson's equation

$$\Delta\Phi = 4\pi G\rho \qquad (1)$$

to accelerated frames ($\Delta$ is the Laplacian in 1913 notation) and extending local mass conservation, expressed as an equation of continuity,

$$\nabla\cdot(\rho v) + \frac{\partial\rho}{\partial t} = 0. \qquad (2)$$

The mass density $\rho$ generalizes to an energy-momentum tensor $T^{\mu\nu}$, analogous to hydrodynamics and electromagnetism. In special relativity, Eq. (2) generalizes to $\partial_\rho T^{\rho\sigma} = 0$ for the local conservation of matter and electromagnetic fields. In general relativity this local conservation law generalizes further, through the covariant derivative, into

$$D_\rho T^{\mu\sigma} \equiv \partial_\rho T^{\mu\sigma} + \Gamma^\mu_{\rho\nu} T^{\nu\sigma} + \Gamma^\sigma_{\rho\nu} T^{\mu\nu} = 0. \qquad (3)$$

Part 2 concluded by noting that the $\Gamma T$ terms gave some interpretation problems to Einstein and his colleagues. Other difficulties were encountered along the way in the collaboration with Marcel Grossmann in 1913. Here we resume the story.

To extend Poisson's equation to a generally covariant tensor equation, the $\Delta\Phi$ term must be replaced with a second-rank tensor. Since gravitational field information would be encoded in metric tensor components, Einstein and Grossmann sought a tensor $\mathsf{G}^{\mu\nu}$, made of $g_{\mu\nu}$ and its derivatives, that would generalize Eq. (1) into an expression of the form

$$\mathsf{G}^{\mu\nu} = \kappa T^{\mu\nu} \qquad (4)$$

where $\kappa$ is a constant proportional to $4\pi G$. In pedagogical treatments this is commonly done by first deriving the equation of a particle in free fall,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\nu}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau} = 0, \qquad (5)$$

a generalization of $\mathbf{a} - \mathbf{g} = \mathbf{0}$. Equation (5) may be derived as the Euler-Lagrange equation resulting from the variational principle $\delta\int (g_{\mu\nu}u^{\mu}u^{\nu})^{1/2}\, d\tau = 0$, where $d\tau$ denotes proper time, and $u^{\mu} \equiv dx^{\mu}/d\tau$. To go to the Newtonian limit, in Eq. (5) one neglects spatial velocities, approximates $d\tau \approx dt$, and considers the effects of gravity as a small perturbation on flat spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad (6)$$

where $|h_{\mu\nu}| << 1$. In this way one shows that $g_{00} \approx 1 + 2\Phi/c^2$, turning Eq. (1) into $\Delta g_{00} = 8\pi G\rho/c^2$, a tentative step toward Eq. (4). However, Einstein did not derive Eq. (5) until 1914,[2] so he and Grossmann had difficulty connecting their work to the Newtonian limit. They also ran into two sticking points that threw the quest off the rails for two years.[3]

The metric tensor components $g_{\mu\nu}$ take the place of $\Phi$, and covariant derivatives take the place of ordinary derivatives. There was a serious problem: the covariant derivative of the metric tensor vanishes identically! Einstein concluded, erroneously, "It seems to follow that the sought-for equations will be covariant only with respect to a certain group of transformations…which for the time being is unknown to us." Grossmann also drew an incorrect argument. He recognized that $\mathsf{G}^{\mu\nu}$ might be the Ricci tensor $R^{\mu\nu}$, but not having a clear path to the Newtonian limit, argued that $R^{\mu\nu}$ "does not reduce to $\Delta\Phi$ in the special case of the weak gravitational field."[4] Reluctantly, the collaborators concluded that covariance would work only when the coordinate transformations were linear in the old coordinates. For if that were so, then to first order in $h_{\mu\nu}$ the $\partial\Gamma \sim \partial(g\partial g)$ terms in the Ricci reduce to the d'Alembertian $\partial^{\mu}\partial_{\mu}h_{\rho\sigma}$ in the weak-field limit.

Seeking necessity in the frustration of not yet achieving his goal of *general* covariance, Einstein offered a physical argument for its nonexistence. This argument, though incorrect, was important because it correctly revealed that $\mathsf{G}^{\mu\nu} = \kappa T^{\mu\nu}$ could *not* determine $g_{\mu\nu}$ *uniquely*. His argument went like this: Divide spacetime into two parts, A and B. Locate the source of $T^{\mu\nu}$ entirely within A. But the source in A also determines the $g_{\mu\nu}$ in B. Now make a coordinate transformation such that $x'_{\mu} = x_{\mu}$ in A but $x'_{\mu} \neq x_{\mu}$ throughout B. Then $g'_{\mu\nu} \neq g_{\mu\nu}$ throughout all of B, even though $T^{\mu\nu}$ has not changed anywhere. Consequently, $g_{\mu\nu}$ is not uniquely determined from the field equations, and therefore (Einstein concluded), *general* covariance is impossible.

This was a great disappointment. Einstein wrote to H. A. Lorentz in August 1913, "Thus, if not all systems of equations of the theory…admit transformations other than linear ones, then the theory contradicts its own starting point [and] all is up in the air."[5]

Einstein was being led astray by his assumption that $\mathsf{G}^{\mu\nu} = \kappa T^{\mu\nu}$ determines the metric tensor uniquely. Later it was realized that he and Grossmann had found, in gravitation, local gauge invariance, which is more familiar in electrodynamics. Maxwell's equations do not determine the potentials $V$ and $\mathbf{A}$ uniquely, so problems in, say, radiation, require "fixing the gauge" by choosing a specific expression for the divergence of $\mathbf{A}$. Similarly, for gravitation, $g_{\mu\nu}$ can be determined by the field equations only to within a transformation $g_{\mu\nu} \rightarrow g'_{\mu\nu}$ wrought by a coordinate transformation $x^{\mu} \rightarrow x'^{\mu}$. For example, to answer Grossmann's objection and show that $R^{\mu\nu}$ *does* reduce to $\Delta\Phi$ in the static, weak-field limit, it's necessary to transform to a coordinate system where $\partial_{\nu}h'^{\mu\nu} = 0$.[6]

Einstein's underlying difficulty at the time was that he did not yet know of a set of constraints on the Riemann tensor, the Bianchi identities. One version states

$$D_{\nu}(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R^{\lambda}_{\lambda}) = 0. \qquad (7)$$

Because any $g_{\mu\nu}$ that solves Eq. (4) must also satisfy Eq. (7), the field equations, *by themselves*, cannot determine the metric tensor components uniquely.

## The Final Stretch

Later in 1913, in a one-semester collaboration with postdoc Adriaan Fokker, Einstein and Fokker published a paper[7] in which they parameterized the metric by introducing a function $\psi$ such that $g_{\mu\nu} = \psi^2\eta_{\mu\nu}$. This yielded an equation of the form $R^{\lambda}_{\lambda} = const.\times T^{\lambda}_{\lambda}$, suggestively reproducing the field equations of the competing gravitational theory of Gunnar Nordström (although Nordström's theory did not predict gravitational deflection of light rays). Thus the Ricci tensor $R^{\mu\nu}$ and $g^{\mu\nu}R^{\lambda}_{\lambda}$ might both have to be included in the final form of $\mathsf{G}^{\mu\nu}$. The Einstein-Fokker paper also corrected an error in the Einstein-Grossmann paper by showing the Ricci tensor *could* give the correct Newtonian limit. The paper closed with the remark, "It is plausible that the role which the Riemann-Christoffel tensor [$R^{\lambda}_{\mu\nu\rho}$] plays in the present investigation would also open a way for the derivation of the Einstein-Grossmann gravitation equations in a way independent of physical assumptions."

It would take two more years for Einstein to close the deal on a generally covariant theory of gravitation. In 1914 he left Zürich for Berlin, and that same year his family life was disrupted when he and Mileva separated. Upon his arrival in Berlin, in the fall of 1914 Einstein presented a paper to the Prussian Academy of Sciences[8] that contained a systematic review of gravitational results to date, including an introduction to tensor calculus and a derivation of Eq. (5), which was shown to produce the Newtonian limit of Eq. (1). He also showed the tensor theory contained the same results as the *c*-field scalar theory of 1911, in particular, the gravitational redshift and the deflection of a light ray grazing the Sun, for which he still calculated 0.83´´. The 1914 paper caught the attention of Tullio Levi-Civita, who graciously corrected some errors, which Einstein gratefully acknowledged.[9] In a letter of January 7, 1915, he wrote a friend, "I firmly believe that the road taken is in principle the correct one…."[10] That spring he took a break from gravitation to work on some magnetism problems with the visiting Johannes de Haas,[11] which resulted in the Einstein–de Haas effect, where a suspended iron cylinder, abruptly magnetized, experiences a torque. In the summer Einstein returned to gravitation.

At the end of June and in the beginning of July 1915, Einstein visited Göttingen University, at the invitation of David Hilbert, to deliver a set of lectures on the state of gravitation theory.[12] General relativity electrified Hilbert and Felix Klein, who tore into it with gusto. Meanwhile, Einstein returned to Berlin and worked through several remaining difficulties by November, writing of himself in the third person that December to Paul Ehrenfest, "That fellow Einstein suits his convenience. Every year he retracts what he wrote the year before."[13] Einstein and

Grossmann had reluctantly assumed general covariance held only for linear transformations, in order to maintain the assumption that the field equations determine the metric tensor components uniquely. Sometime in the fall of 1915 he shook himself free of that misconception. On October 12 he wrote to Lorentz, "In my paper of [October 1914], I carelessly introduced the assumption that [the gravitational Lagrangian] is an invariant for linear transformations," a restriction he removed in a paper of November 4.[14] On November 7 he wrote to Hilbert, "I realized about four weeks ago that my methods of proof used until then were deceptive."[15]

Lacking the final form of the Lagrangian, he managed to construct the correct field equations another way. He began with a superposition of the two rank-two tensors based on $R^{\lambda}{}_{\mu\nu\rho}$ that could make up $G^{\mu\nu}$:

$$AR^{\mu\nu} + Bg^{\mu\nu}R^{\lambda}{}_{\lambda} = \kappa T^{\mu\nu} \qquad (8)$$

where $A$ and $B$ are constants to be determined by the constraints of the covariant conservation law $D_{\mu}T^{\mu\nu} = 0$, and by the Newtonian limit. In pedagogical treatments nowadays one takes the covariant derivative of Eq. (8) and notes that $D_{\mu}T^{\mu\nu} = 0$, which triggers the Bianchi identity of Eq. (7) and immediately gives $B = -\frac{1}{2}A$. The Newtonian limit requires $A = -1$,[16] resulting in the Einstein field equations as we know them:[17]

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R^{\lambda}{}_{\lambda} = -\kappa T^{\mu\nu} \qquad (9)$$

where $\kappa = 8\pi G$ (in units where $c = 1$; in SI units $\kappa = 8\pi G/c^4$). However, even in 1915 Einstein was still not aware of the Bianchi identities.[18] So he worked around them by choosing coordinates where $|\det g_{\mu\nu}| = 1$ and invoked $D_{\mu}T^{\mu\nu} = 0$.[19] In this way he obtained the correct field equations. He knew them to be correct because they predicted the observed anomalous precession of Mercury's orbit, a discovery which brought him great joy, as this anomaly had been an outstanding unsolved problem for 60 years. The redshift prediction survived intact, but in recalculating the light deflection, this time he obtained 1.75′′, about twice his former value and, crucially, quite distinct from the Newtonian prediction by about a factor of 2.

When Einstein obtained the final form of the gravitational field equations, Hilbert was close behind him, making a significant contribution to general relativity by coming up with the Lagrangian density, whose Euler-Lagrange equation yields those field equations. Along the way, Hilbert and Klein (along with Einstein) worried about energy conservation in general relativity. Hilbert made a distinction between "proper" and "improper" conservation laws. A proper conservation law has a clean equation of continuity, like Eq. (2) or its generalization to $\partial_{\rho}T^{\rho\sigma} = 0$. If $\partial_{\rho}T^{\rho\sigma} \neq 0$, the quantity represented by $T^{\rho\sigma}$ is *not* conserved.[20] In the fall of 1915, Hilbert and Klein asked mathematician Emmy Noether, an expert in invariance theory, to assist them in trying to understand improper energy conservation in the context of general relativity. In response, Noether developed a general theorem relating conservation laws to symmetries, with applications that sweep across all of physics. There are actually two Noether theorems; the second extends the first.[21, 22] The first applies to global spacetime transformations and global gauge invariance. If a system is invariant under global spacetime transformations, then energy and momentum are conserved; invariance under global gauge transformations gives charge conservation. The second theorem considers local gauge invariance. Applied to Hilbert's Lagrangian, Noether's second theorem shows the Bianchi identities to emerge as a necessary consequence, and from that follows the conservation of energy in the form of an equation of continuity with the covariant derivative, Eq. (3).

According to Noether's second theorem, conservation laws in general relativity are *necessarily* improper. The energy of the gravitational field, by itself, is *not* conserved, because gravity and matter exchange energy. A proper conservation law for energy emerges only by enclosing the *entire* system, all matter and fields, with a surface at infinity. Out there, spacetime asymptotically becomes Minkowskian, and the surface integral of the covariant divergence goes over to the surface integral of a proper divergence. Only in that sense does general relativity contain a proper energy conservation law.

In early 1916, Einstein published the first written account of the completed theory of general relativity.[23] His applications of the new theory were done with perturbation theory, along the lines of Eq. (6). But on January 16, 1916, he read a significant paper to the Prussian Academy of Sciences on behalf of Karl Schwarzschild, who was in the German army at the Russian front. Schwarzschild had found the first exact solution to Einstein's equations, the $g_{\mu\nu}$ in the spacetime around a static, uncharged point mass $M$.[24]

In 1919 a British expedition took the eclipse photographs that measured the solar deflection of a light ray. The Sobral site yielded 1.98 ± 0.16′′, the Principe site 1.68 ± 0.40′′, both ruling out the Newtonian prediction and showing close agreement with Einstein's 1.75′′ prediction of 1915.[25] This measurement had tremendous social impact, too. The year after the close of World War I, a British expedition tested the calculation of a scientist in Germany. A public weary of mustard gas and slaughter had a transcendent moment of shared humanity directed toward higher things. Einstein became an instant celebrity, a role he never wanted. But he used his fame well for the rest of his life in promoting causes of social justice, equality, and tolerance.[26]

Einstein started down the road to general relativity in 1907. The journey took until the end of 1915 before reaching a successful destination. Along the way, he began with crude approximations, took side journeys into other projects, questioned his assumptions, learned a new branch of mathematics, engaged collaborators for help on sticking points, lived life with a family through times of joy and distress, persevered through multiple crises and setbacks, worked with intense focus, and, in the final years, exhausted himself while overcoming enormous struggles. After it was all over, on June 20, 1933, in a lecture at the University of Glasgow, Albert Einstein, creator of the general theory of relativity, recalled the long and winding road:

*The years of searching in the dark for a truth that one feels but cannot express, the intense desire and the alternations of confidence and misgiving until one breaks through to clarity and understanding, are known only to him who has himself experienced them.*[27]

May general relativity's centennial inspire each of us to bring our personal best to the tasks we face! **Ω**

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[3] *Ibid*., 221–222.
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[5] Pais, ref. 2, 228.
[6] *Ibid*., 222–223.
[7] *Ibid*., 236–237; Albert Einstein and A. D. Fokker, *AdP* **44**, 96 (1913).
[8] Pais, ref. 2, 243.
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[12] C. Reid, *Hilbert* (Springer-Verlag, 1972).
[13] Pais, ref. 2, 250.
[14] *Ibid*., 250.
[15] *Ibid*., 259.
[16] See, e.g., Neuenschwander, *Tensor Calculus for Physics* (Johns Hopkins University Press, 2015), 150.
[17] If the cosmological constant $\Lambda$ is included, the field equation reads $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R^{\lambda}{}_{\lambda} - g^{\mu\nu}\Lambda = -\kappa T^{\mu\nu}$. But $\Lambda$ did not appear until Einstein applied the equations to the closed universe of cosmology in 1917. See "History of Big Bang Cosmology, Part 2: The Problem with Infinity," *Radiations*, Spring 2008, pp. 25–29.
[18] Hilbert was not yet aware of the Bianchi identities either, according to Pais, ref. 2, 258. Tensor calculus was still relatively new at that time.
[19] Pais, ref. 2, 256. See also Robert E. Kennedy, *A Student's Guide to Einstein's Major Papers* (Oxford University Press, 2012), ch. 5.
[20] A familiar example is found in Poynting's theorem of electrodynamics, div $\mathbf{S} + \partial\eta/\partial t = -\mathbf{j}\cdot\mathbf{E}$, where $\mathbf{E}$ is the electric field, $\mathbf{j}$ the electric current density, $\eta$ the electromagnetic energy density, and $\mathbf{S}$ is Poynting's vector. If $\mathbf{j}\cdot\mathbf{E} = 0$, then Poynting's theorem becomes a proper equation of continuity for electromagnetic field energy; if $\mathbf{j}\cdot\mathbf{E} \neq 0$, then the electromagnetic field energy is not conserved because the fields and charged matter exchange energy. The energy of the coupled matter-field system is conserved, for integrating Poynting's theorem over all space says that the energy lost by the electromagnetic field equals the work done on matter by the electric field. In Poynting's theorem, the $\mathbf{j}\cdot\mathbf{E}$ term plays a role analogous to the $\Gamma T$ terms in Eq. (3).
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