Fall 2012 *Radiations* Puzzle Corner Physics Problem Answers and Solutions

I. Answers

   A. $\Delta U = 1.98 \times 10^{20}$ J
   B. $\Delta T = -9.92 \times 10^{19}$ J
   C. $\Delta T_m = 2.72 \times 10^{21}$ J
   D. $\Delta t = 2.10 \times 10^{-7}$ °C

II. Solutions

   On the next and following pages
The Moon orbits the Earth, held in approximately circular motion. Due to the variation of the Moon’s gravitational force with distance, tidal bulges form on the surface of the Earth’s oceans. Frictional forces carry the bulges ahead of the Earth – Moon line. These bulges apply gravitational torques on the Moon, increasing the angular momentum of the Moon around the Earth, increasing its orbital energy, raising it to higher orbits, and decreasing the angular momentum of the Earth about its axis.

Mass of the Earth: \( M = 5.97 \times 10^{24} \text{ kg} \)

Mass of the Moon: \( m = 7.35 \times 10^{22} \text{ kg} \)

Radius of the Moon’s orbit: \( r \)

Speed of the Moon: \( v \)

Initial Radius of the Moon: \( r_0 = 3.84 \times 10^8 \text{ m} \)

Increase of Moon’s orbit per century: \( \Delta r = 1 \text{ m} \)

A. Find: \( \Delta U \), the increase in the Potential Energy of the Moon as it orbits the Earth for a century.

\[
\Delta U = U_f - U_0; \quad \text{Where} \quad U = -\frac{GMm}{r} \quad \text{and} \quad G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2
\]

In principle, one could calculate \( U \) at the final and initial radii and subtract, but that requires great numerical precision. We can find a more accurate quantity by finding a good approximate expression in terms of \( \Delta r \).

\[
\Delta U = \frac{-GMm}{r_f} - \frac{-GMm}{r_0}
\]

\[
\Delta U = -GMm \left( \frac{1}{r_0 + \Delta r} - \frac{1}{r_0} \right)
\]

\[
\Delta U = -GMm \left( \frac{1}{1 + \Delta r / r_0} - 1 \right)
\]

\[
\Delta U = -GMm \left( \left(1 + \Delta r / r_0\right)^{-1} - 1 \right); \quad \text{for} \quad x << 1, (1 + x)^n \approx 1 + nx
\]

\[
\Delta U = -GMm \left(1 - \Delta r / r_0 - 1 \right)
\]
\[ \Delta U = \frac{GMm}{r_0^2} \Delta r; \text{This may also be understood as} \]

\[ \Delta U = -F \Delta r, \text{where } F = -\frac{GMm}{r^2} \]

The force of gravity between the Earth and Moon

We may now insert the values of these quantities to obtain

\[ \Delta U = 1.98 \times 10^{20} \text{ J.} \]

B. Find: \( \Delta T \), the kinetic energy lost over the course of a century.

To this end, it would be useful to express the kinetic energy as a function of \( r \).

\[ T = \frac{1}{2} mv^2 \]

To relate \( v \) to \( r \), consider the centripetal force

\[ F = ma \]

\[ -\frac{GMm}{r^2} = m \left( -\frac{v^2}{r} \right) \]

\[ mv^2 = G \frac{Mm}{r} \]

This is very close in form to the potential energy,

\[ U = -\frac{GMm}{r} \]

\[ mv^2 = -U \]

\[ T = -\frac{1}{2} U \]

\[ \Delta T = -\frac{1}{2} \Delta U \]

\[ \Delta T = -\frac{1}{2} \left( 1.98 \times 10^{20} \text{ J} \right) \]

\[ \Delta T = -9.92 \times 10^{19} \text{ J} \]

C. Find: \( \Delta T_e \), the change in the Earth’s rotational kinetic energy.

\[ T_m = \frac{1}{2} I_e \omega_e^2, \text{ where } \omega_e = \frac{2\pi}{T} \text{ is the Earth's Angular Velocity about its axis,} \]

which roughly corresponds to the axis of the Moon's orbit.

\[ T = 24 \text{ hr} \text{ is the Period of the Earth's rotation.} \]

\[ I_e \text{ is the Moment of Inertia of the Earth.} \]
If we model the Earth as a Uniform Solid Sphere,

\[ I_e = \frac{2}{5} MR^2, \] where \( M \) is the mass of the Earth,

\[ R = 6.378 \times 10^6 \text{ m} \] is the radius of the Earth.

\( L_e = I_e \omega_e \) is the Rotational Angular Momentum of the Earth.

\[ T_e = \frac{L_e^2}{2I_e} \]

\[ \Delta T_e = T_e - T_{e0} \]

\[ \Delta T_e = \frac{L_{e0}^2 - L_{e0}^2}{2I_e} \]

\[ \Delta T_e = \frac{1}{2I_e} \left[ \left( (L_{e0} + \Delta L_e)^2 - L_{e0}^2 \right) \right] \]

\[ \Delta T_m = \frac{1}{2I_e} \left[ L_{e0}^2 + 2L_{e0} \Delta L_e + (\Delta L_e)^2 - L_{e0}^2 \right], \text{ and since } \Delta L_e \ll L_{e0} \]

\[ \Delta T_m = \frac{L_{e0}^2}{I_e} \Delta L_e \]

\[ \Delta T_e = \omega_e \Delta L_e, \text{ so the exact value of } I_e \text{ is not needed} \]

To calculate \( \Delta L_e \), one may invoke Conservation of Angular Momentum, which applies to the Earth-Moon system considered as an isolated system.

\[ \Delta L = 0 \]

\[ \Delta \left( L_e + L_m \right) = 0, \text{ where } L_m = mvr \text{ is the Orbital Angular Momentum of the Moon,} \]

\[ \Delta L_e = -\Delta L_m \]

\[ \Delta T_m = -\omega_e \Delta L_m \]

It might be useful to express \( L_m \) in terms of \( r \).

\[ L_m = mvr \]

\[ T = -\frac{1}{2} U, \text{ as we demonstrated earlier in B.} \]

\[ \frac{1}{2} mv^2 = -\frac{1}{2} \left( -\frac{GMm}{r} \right) \]

\[ v = \sqrt{\frac{GM}{r}} \]

\[ L_m = m \sqrt{\frac{GM}{r}} \]

\[ L_m = m \sqrt{GM} r^{1/2} \]

\[ \Delta T_e = -\omega_e \left( L_{mf} - L_{m0} \right) \]

\[ \Delta T_e = -\omega_e \left( m \sqrt{GM} r^{1/2} - m \sqrt{GM} r_0^{1/2} \right) \]
\[ \Delta T = -\omega e m \sqrt{GM} \left( r_0 + \Delta r \right)^{\frac{1}{2}} - r_0^{\frac{1}{2}} \]  

\[ \Delta T = -\omega e m \sqrt{GM} r_0^{\frac{1}{2}} \left( 1 + \frac{\Delta r}{r_0} \right)^{\frac{1}{2}} - 1, \] and if we remember \( (1 + x)^n \approx 1 + nx \)

\[ \Delta T = -\omega e m \sqrt{GM} r_0^{\frac{1}{2}} \left( 1 + \frac{1}{2} \frac{\Delta r}{r_0} - 1 \right) \]

\[ \Delta T = -\omega e \frac{m}{2} \sqrt{\frac{GM}{r_0}} \Delta r \]

\[ \Delta T_e = -2.72 \times 10^{21} \text{ J} \]

D. As the ocean tidal bulges are dragged along with the Earth by friction, they slip. This friction converts mechanical energy to thermal energy. This heat flow, \( Q \), into the Earth (including the ocean) will raise the temperature of the Earth. One may approximate the specific heat of the Earth as one half that of water.

Find: \( \Delta t \), the temperature rise of the Earth over a century due to this effect.

\[ Q = C \Delta t, \text{ where } C \text{ is the heat capacity of the object} \]

\[ \Delta t = \frac{Q}{C} \]

\[ \Delta t = \frac{Q}{M c}, \text{ where } c \text{ is the specific heat of the substance} \]

\[ \Delta t = \frac{Q}{M \frac{1}{2} c_w}, \text{ where } c_w = 4.186 \frac{J}{kg \cdot ^\circ C}, \text{ the specific heat of water} \]

The heat flow, \( Q \), may be inferred from Conservation of Energy

\[ \Delta E = 0 \]

\[ \Delta T + \Delta U + \Delta T_m + Q = 0 \]

\[ Q = -(\Delta T + \Delta U + \Delta T_m) \]

\[ \Delta t = -\frac{2(\Delta T + \Delta U + \Delta T_m)}{M c_w} \]

When we insert the values of energy lost and gained from parts A., B., and C., along with the values of the Mass of the Earth the specific heat of water that are reproduced in these solutions, one obtains:

\[ \Delta t = 2.10 \times 10^{-7} ^\circ C \]