

# History of Big Bang Cosmology, Part 2: The Problem with Infinity

## ELEGANT CONNECTIONS IN PHYSICS

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### 1. THE PROBLEM WITH INFINITY

This is the second stop on a tour of some high points in the history of big bang cosmology. Part 1, “Cosmic Geography,” described how spiral nebulae came to be understood as galaxies, with the advent of “standard candles” to measure cosmic distances.[1]

When Albert Einstein completed the General Theory of Relativity (GR) in 1915, there were few opportunities to test the theory. Nevertheless, in 1917, Einstein boldly applied GR to the *entire universe*, for GR offered hope for the resolution of a seemingly irreconcilable conflict between Newtonian gravitation and cosmology. The sticking point was the boundary condition at infinity.

Newtonian gravitation tells us how to find the gravitational potential  $\Phi$  produced by a mass distribution. We solve Poisson’s equation,

$$\nabla^2 \Phi = 4\pi G\rho \quad (1)$$

where  $\rho$  denotes the source mass density, and  $G = 6.67 \times 10^{-11}$  Nm<sup>2</sup>/kg<sup>2</sup> is Newton’s gravitational constant. If the potential vanishes at infinity, the solution of Eq. (1) is

$$\Phi(\mathbf{r}) = -G \int d^3r' \rho(\mathbf{r}') / |\mathbf{r} - \mathbf{r}'| \quad (2)$$

where the integral extends over all space, and  $\mathbf{r} - \mathbf{r}'$  denotes the vector from the source point  $\mathbf{r}'$  to the field point  $\mathbf{r}$ . For the integral to converge, the *density must vanish faster than inverse square at infinity*.

The other side of Newton’s program tells us how a test particle free-falls in response to the local gravitational field. The trajectory of a particle of inertial mass  $m$  is determined by Newton’s Second Law,  $\mathbf{F} = m d^2\mathbf{r}/dt^2$ . The gravitational force on the response particle is  $m^*\mathbf{g}$ , where  $\mathbf{g} = -\nabla\Phi$  and  $m^*$  denotes the response particle’s “gravitational charge.” In Einstein’s time  $m^*/m$  was known to be the same for all bodies to a few parts in a hundred million; postulating the equivalence to be exact, one can measure inertial mass and gravitational mass in the same units and set  $m^*/m$  equal to unity.[2] Newtonian theory accepts this equivalence as a coincidence, and cancels the masses to find

$$d^2\mathbf{r}/dt^2 - \mathbf{g} = 0, \quad (3)$$

consistent with the observation that any two particles at the same location free-fall with the same acceleration. The successes of this paradigm on the solar system and galactic scales

gave Newtonian cosmologists reason to turn with confidence to entire universe.

Gravity only attracts, yet the universe has not collapsed. To explain the non-occurrence of this catastrophe, Newton hypothesized that the star population extends to infinity, symmetrically in all directions, with uniform density at the cosmic scale. However, if the density  $\rho$  does not vanish at infinity, then the integral of Eq. (2) cannot converge! By 1917, with his shiny new GR in hand, Einstein could suggest a way out of this long-standing impasse.

### 2. GRAVITATION ACCORDING TO EINSTEIN

Einstein conceptualized gravity as a massive body “curving the spacetime” about it. Recall from Special Relativity that the *proper time*  $d\tau$  between two nearby events in spacetime is related to the lab time  $dt$  (measured in meters, using the speed of light to convert between meters and seconds) and the lab distance  $ds$ , through the difference in squares:[3]

$$d\tau^2 \equiv dt^2 - ds^2. \quad (4)$$

With spatial displacements expressed in spherical coordinates,  $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \equiv dr^2 + r^2 d\Omega^2$ . The numerical value of  $d\tau$  forms an invariant among all the inertial frames of Special Relativity.

By restricting itself to inertial frames, Special Relativity entertains only “flat” spacetimes with zero curvature. But in GR we allow *accelerated* frames. Because of the Principle of Equivalence, an accelerated frame is *locally* indistinguishable from gravitation; hence, GR is a theory of gravity. Summing over repeated indices, in arbitrary spacetimes the Interval generalizes to

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (5)$$

where the  $dx^\mu$  denote *coordinate* displacements, the superscripts labeling the various spacetime coordinates  $x^\mu$ ; e.g., in Eq. (4) with spherical coordinates we take  $x^0 = t$ ,  $x^1 = r$ ,  $x^2 = \theta$ ,  $x^3 = \phi$ . The quantities  $\{g_{\mu\nu}\}$  are the components of the metric tensor for spacetime, which convert *coordinate* displacements into *distances*. In Eq. (5) the nonzero  $g_{\mu\nu}$  are  $g_{00} = 1$ ,  $g_{11} = -1$ ,  $g_{22} = -r^2$ ,  $g_{33} = -r^2 \sin^2\theta$ . Einstein’s gravitational field equations [which replace Eq. (1)] must provide a way for us to *find* the metric tensor components when a gravity source is given. Conversely, Einstein’s gravitational kine-

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matics [that replaces Eq. (3)] must describe how a particle free-falls in a spacetime “distorted” by gravity. [4]

To visualize what the “curvature of spacetime” means, imagine a two-dimensional surface mapped with polar coordinates  $(r, \varphi)$ . Paint a set of concentric rings centered on the origin. When circumnavigating the origin on one ring, the odometer on your motorcycle measures a circumference  $C$ . This operation allows us to *define* the  $r$ -coordinate as  $r \equiv C/2\pi$ . Different rings may be marked with signage announcing their  $r$ -coordinates. Now suppose you ride your motorcycle radially from the ring marked  $r = 17\text{km}$  to the one marked  $r = 18\text{km}$ . You are astonished to see that the calibrated odometer on your motorcycle records not 1 km, but 1.23km! If you are not too set in your geometrical ways, you might hypothesize that the metric tensor features a parameter  $a$  such that, for example,[5]

$$ds^2 \equiv (1 - a/r)^{-1} dr^2 + r^2 d\varphi^2 \quad (6)$$

for  $r > a$ . How could one *measure*  $a$ ? When riding radially ( $d\varphi = 0$ ) from  $r = A$  to  $r = B$  the odometer measures  $\Delta s$ , and Eq. (10) says that

$$\Delta s = \int_A^B (1 - a/r)^{-1/2} dr \quad (7)$$

which integrates to  $\Delta s = [B(B-a)]^{1/2} - [A(A-a)]^{1/2} + a \ln [f(B)/f(A)]$ , where  $f(X) \equiv X^{1/2} - (X-a)^{1/2}$ . Your odometer data for the trip from  $r = 17\text{km}$  to  $r = 18\text{km}$  infers that  $a = 6\text{km}$ . We see that  $\Delta s$  will equal  $\Delta r$  for radial motion only when  $a = 0$ . In general, a local radius of curvature may be computed through a “curvature tensor,” that involves derivatives of the metric tensor. Its trace gives the local curvature  $k$ . In our example,  $k = \frac{1}{2} (a/r^3) (1 - a/r)$  for  $r > a$ . [6]

In Einstein’s program, the equation that takes over the job of Poisson’s equation is actually a set of *ten* equations called “Einstein’s field equations,” [7]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \quad (8)$$

where each subscript ranges over the indices of the four spacetime coordinates. The Laplacian of Eq. (1) has been replaced with the components of the Ricci curvature tensor  $R_{\mu\nu}$  that is linear in the second derivatives of the metric tensor, and with its trace  $R$ , which measures the local curvature of the spacetime. The density term in Poisson’s equation has been replaced with the “stress tensor”  $T_{\mu\nu}$  that counts up all forms of mass-energy that cause gravity. For example, the time-time component,  $T_{00}$ , is typically the energy density of matter and radiation.

Einstein’s extension of Eq. (3) describes the motion of a particle in free-fall according to

$$d^2 x^\mu / dt^2 + \Gamma^\mu_{\nu\sigma} u^\nu u^\sigma = 0 \quad (9)$$

where the  $u^\nu$  are coordinate velocities with respect to proper time, and the coefficients  $\Gamma^\mu_{\nu\sigma}$  are called the “affine connec-

tion” coefficients or “Christoffel symbols.” [8] Through derivatives they relate the locally flat spacetime coordinates with the global curvilinear coordinates. As shown in the textbooks, the affine connection coefficients may also be written exclusively in terms of the gradients of the metric tensor of the global curvilinear coordinates. The  $\Gamma^\mu_{\nu\sigma}$  depend not only on the intrinsic properties of the space, but also on the coordinate system; they may be nonzero in a flat space described in curvilinear coordinates (although the curvature tensor  $R_{\mu\nu}$  vanishes in flat space, whatever the coordinate system). We can see by comparing Eqs. (3) and (19) that the  $\Gamma^\mu_{\nu\sigma}$  are *analogous* to the components of  $\mathbf{g}$ . Furthermore, because the  $\Gamma^\mu_{\nu\sigma}$  involve gradients of the metric tensor, the  $g_{\mu\nu}$  were called “gravitational potentials” by Einstein.

Einstein *could* have included on the left-hand side of Eq. (8) another two-index tensor of the form  $g_{\mu\nu}\Lambda$ , where  $\Lambda$  is a *constant* quite unrelated to curvature. In deriving his field equations, there are not enough constraints to *determine*  $\Lambda$ ; if you want  $\Lambda$  you have to put its numerical value in by hand from empirical data. While the  $\Lambda g_{\mu\nu}$  term is *logically* permissible, because Einstein was initially committed to his elegant vision of “gravity as curvature,” he set  $\Lambda = 0$ . And  $\Lambda$  was not necessary on the solar system scale anyway. As we are about to see, Einstein was soon obliged to re-visit this decision.

### 3. EINSTEIN’S COSMOLOGY OF 1917

In 1917 Einstein applied GR to cosmology, and started the discussion that led over the next dozen years, through the work of others, to big bang cosmology. The title of his seminal paper translates to “Cosmological Considerations on the General Theory of Relativity,” and was published in the *Proceedings of the Prussian Academy of Science* in 1917. [9]

The paper begins by recalling the old problem at infinity, and turns to the possibilities for its resolution in GR. Curiously, Einstein begins this part of the discussion with comments that are unusual in the archival journals: he describes the *journey* as well as the *destination*. He writes,

*In the present paragraph I shall conduct the reader over the road that I have myself travelled, rather a rough and winding road, because I cannot hope otherwise that he will take much interest in the result at the end of the journey. The conclusion I shall arrive at is that the field equations of gravitation which I have championed hitherto still need a slight modification, so that on the basis of the general theory of relativity those fundamental difficulties may be avoided which have been set forth in § 1 as confronting the Newtonian theory.*

Einstein’s first attempt to reconcile GR with infinity assumed that the concept of inertial mass requires one body to be compared to *another body*. Thus the inertia of an *isolated* mass must be zero. Assuming a homogeneous and isotropic universe, Einstein introduced this parameterization of the metric,

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$$d\tau^2 = Bdt^2 - A(dx^2 + dy^2 + dz^2) \quad (10)$$

where  $A$  and  $B$  depend on location. A free particle of mass  $m$  carries momentum  $p = mds/d\tau = mv\sqrt{A(dt/d\tau)}$  where  $v = ds/dt$  and, from the Interval,  $(dt/d\tau) = (B - Av^2)^{-1/2}$ . To first order in  $v$  one obtains  $p \approx mv\sqrt{A/B}$ , so that  $m\sqrt{A/B}$  serves as an effective mass. If the inertial mass of a body isolated at infinity must vanish, then  $A \rightarrow 0$  or  $B \rightarrow \infty$ ; either way the metric at infinity becomes nonsensical. Einstein made two other attempts to salvage the situation with unmodified GR.

(a) One could *impose* the Minkowski metric at infinity. This one does when solving problems at the stellar scale, where “infinity” means the space *between* stars. But Einstein was reluctant to foist such boundary condition on the *entire universe* because such an imposition would be “contrary to the spirit of the relativity principle,” and disagrees with the relativity of inertia as given above.

(b) Another possibility was to  
*...refrain entirely from laying down boundary conditions for spatial infinity claiming general validity; but at the spatial limit of the domain under consideration we have to give the  $g_{\mu\nu}$  separately in each individual case...The possibility (b) holds out no hope of solving the problem, but amounts to giving it up...*

Albert Einstein was not known for giving up. A resourceful fellow, he now tries something radical:

*...Nevertheless, there is still a possible way out, without resigning as suggested under (b). For if it were possible to regard the universe as a continuum which is finite (closed) with respect to its spatial dimensions, we should have no need at all of any such boundary conditions.*

If mass curves spacetime, it was conceivable that the matter of the entire universe might curve *space* back onto itself to form a finite, unbounded geometry. With no infinity, the problem at infinity dissolves! This revolutionary postulate was supported by no empirical evidence; it was a bold guess.

*We shall proceed to show that both the general postulate of relativity and the fact of the small stellar velocities are compatible with the hypothesis of a spatially finite universe; though certainly, in order to carry through this idea, we need a generalizing modification of the field equations of gravitation.*

The last sentence indicates that Einstein’s frustration was not over: the closed universe was not consistent with GR until he modified his original equations by including the cosmological constant.

Let’s investigate the metric tensor we would expect from a model universe *constructed* to close back on itself like the surface of a sphere. Then we’ll compare that result to the metric that follows from solving Einstein’s field equations applied to his model of the universe.

Einstein imagined the *three-dimensional physical space* to be the *surface* of a sphere that exists in a *four-dimensional* Euclidian space. Any point in the physical universe has three-

dimensional coordinates  $(x, y, z)$  that maps to a point on the *surface* of this hypersphere. Map the four spatial dimensions with rectangular coordinates  $(x, y, z, u)$ . In this world mark out a sphere of radius  $a$  centered on an origin that, by homogeneity, can be located anywhere. That surface is described by the equation

$$a^2 = r^2 + u^2 \quad (11)$$

where  $r^2 = x^2 + y^2 + z^2$ . Using spherical coordinates for the three familiar spatial dimensions, the spacetime interval between any two events becomes

$$d\tau^2 = dt^2 - du^2 - dr^2 - r^2 d\Omega^2. \quad (12)$$

Any journey in the physical universe lies on the *surface* of the hypersphere, so that  $du = -rdr/u = rdr/(1 - r^2/a^2)^{1/2}$ , which turns Eq. (12) into

$$d\tau^2 = dt^2 - (1 - r^2/a^2)^{-1} dr^2 - r^2 d\Omega^2. \quad (13)$$

We *assumed* a spherical geometry of radius  $a$ , which requires  $g_{rr} = -(1 - r^2/a^2)^{-1}$ . Now let’s reverse the argument and see if this metric emerges as a solution to Einstein’s field equations.

#### 4. METRIC FROM THE FIELD EQUATIONS

When one aspires to find the metric tensor by solving the field equations, one builds into its parameterization whatever symmetries are available. Einstein’s model of the universe was based on two major assumptions. First, at the cosmic scale, the universe appears to be isotropic and homogeneous. Second, few stars had been seen to change their positions relative to others over time; overall the universe seemed static. The assumption of a static universe means that all time derivatives of the metric tensor are set to zero, thereby ruling out the possibility of a *dynamic* model universe. In an isotropic and homogeneous universe, the metric could depart from flat spacetime only through rescalings in the coordinate  $r$  (no angular dependence) of the temporal and radial displacements. Therefore we can parameterize the Interval as

$$d\tau^2 = e^{A(r)} dt^2 - e^{B(r)} dr^2 - r^2 d\Omega^2. \quad (14)$$

which sets our task as finding  $A(r)$  and  $B(r)$ . Jumping across to the right-hand side of the field equations, we need an expression for the stress-energy tensor. An isotropic, homogeneous universe suggests modeling the contents of the universe as a “dust” of stars with uniform mass density  $\rho$ . The light radiated by those stars generates a pressure  $P$ . Borrowing heavily from fluid mechanics, from the density and pressure one constructs the stress tensor components. We now have all the pieces necessary to write out Einstein’s field equations. There happen to be three independent equations: [10]

$$e^{-B(r)} (A' / r + 1/r^2) - 1/r^2 = 8\pi G\rho, \quad (15)$$

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$$e^{-B(r)} (B'/r - 1/r^2) + 1/r^2 = 8\pi G\rho, \quad (16)$$

and

$$P' = -\frac{1}{2}(\rho + P)A' \quad (17)$$

where primes denote derivatives with respect to  $r$ .

An isotropic and homogeneous universe means that the energy density and pressure are uniform throughout space, so  $\rho = \rho_0 = \text{constant}$  and  $P = P_0 = \text{constant}$ . Therefore  $P' = 0$ , so that Eq. (17) requires  $(\rho + P)A' = 0$ , which leads to an interesting decision. Because the density of matter and the radiation pressure are non-negative, Einstein chose  $A' = 0$ , giving  $A = \text{const}$ . Requiring the metric to be Minkowskian as  $r \rightarrow \infty$  sets  $e^A = 1$ , which turns Eq. (15) into

$$e^{-B(r)} = 1 + r^2/a_E^2 \quad (18)$$

where we have introduced the ‘‘Einstein radius’’  $a_E$ ,

$$a_E^2 \equiv (8\pi G P_0)^{-1}.$$

We have found the metric of Einstein’s model:

$$d\tau^2 = dt^2 - (1 + r^2/a_E^2)^{-1} dr^2 - r^2 d\Omega^2. \quad (19)$$

You will notice a crucial difference in sign between the  $g_{rr}$  of Eq. (19) compared to that of Eq. (13). The former has  $g_{rr} = -(1 - r^2/a^2)^{-1}$  for a space *postulated* to close back on itself. But the metric derived from the GR field equations yields  $g_{rr} = -(1 + r^2/a^2)^{-1}$ . What geometry does this describe?

In contrast to a spherical surface embedded in four dimensions, consider a surface in four dimensions whose equation features a sign opposite that of the sphere [compare to Eq. (11)]:

$$a^2 = r^2 - u^2. \quad (20)$$

By similar steps that led us to Eq. (13) we obtain for the hyperbolic spacetime:[11]

$$d\tau^2 = dt^2 - (1 + r^2/a^2)^{-1} dr^2 - r^2 d\Omega^2. \quad (21)$$

describing an *open* space, *not* the geometry of a closed, spherical universe. The solutions to Einstein’s original field equations seem to have led us to a hyperbolic universe, not a spherical one! Hyperbolas extend to infinity, resurrecting the infernal problem of the boundary condition at infinity! Einstein’s only other option within his original model was to set  $\rho_0 + P_0 = 0$ . Because the existence of massive stars requires  $\rho_0 > 0$ , Einstein was led to conclude,

*The objection to this solution is that one has to introduce a negative pressure, for which there exists no physical justification...*

But the confirmation that GR enjoyed in accounting for the anomalous precession of Mercury suggested even in 1917 that the theory was not *wrong*. Perhaps cosmology was merely revealing GR to be *incomplete*. Perhaps GR could be *extended* without reversing its core principles. The price would be to go a step beyond the vision that ‘‘gravitation is curvature.’’ Einstein reluctantly reconsidered his original field equations, and wrote in the 1917 paper,

*...if it were certain that the field equations which I have hitherto employed were the only ones compatible with the postulate of general relativity, we should probably have to conclude that the theory of relativity does not admit the hypothesis of a spatially finite universe.*

*However, the system of equations allows a readily suggested extension which is compatible with the relativity postulate, and is perfectly analogous to [an] extension of Poisson’s equation...*

Earlier in the paper Einstein explored such a scenario with a cosmological constant  $\Lambda$  in Poisson’s equation,

$$\nabla^2\Phi - \Lambda\Phi = 4\pi G\rho. \quad (22)$$

With a  $\Lambda$  sufficiently small, at the scale of a solar system or galaxy the mass density  $\rho$  dominates,  $\Lambda\Phi$  remain negligible, and the familiar Newtonian solution obtains. But at the cosmic scale that sees the entire universe as a dust of galaxies with some smoothed-out constant density  $\rho_0$ ,  $\Phi$  assumes a uniform value  $\Phi_0$  and  $\nabla^2\Phi$  vanishes. Under this circumstance the modified Poisson’s equation fixes the cosmic potential as  $\Phi_0 = -4\pi G \rho_0/\Lambda$ , which gives zero  $\mathbf{g}$  field to allow a static universe. Of course, when Einstein trotted out this modification to Poisson’s equation he was not trying to revive Newtonian cosmology; he was only demonstrating how a new constant could extend gravitation to cosmological scales without spoiling a theory’s established successes.

The GR field equations were constructed with the elegant assumption that *only* the Riemann curvature tensor was necessary to capture the essence of gravitation. But another logical possibility exists with the cosmological constant term  $g_{\mu\nu}\Lambda$ . Einstein followed this possibility despite his reluctance:

*...the introduction of this second member constitutes a complication of the theory, which seriously reduces its logical simplicity.*

With this ‘‘extra’’ term, the field equations become

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}. \quad (23)$$

The term  $\Lambda g_{\mu\nu}$  may also be thought of as a new kind of gravity *source*. Shift it over to the other side of the field equations, which become

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R = -8\pi G (T_{\mu\nu} - g_{\mu\nu} \Lambda/8\pi G). \quad (24)$$

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If we define  $\Lambda > 0$ , to guarantee a *static* universe  $\Lambda$  must *oppose* the attraction of ordinary gravitating sources; hence the sign chosen for  $\Lambda$  in the modified field equation. The cosmological constant introduces an effective energy density and pressure into the universe that are *opposite* in sign:

$$\rho_\Lambda = \Lambda/8\pi G \quad \text{and} \quad P_\Lambda = -\rho_\Lambda. \quad (25)$$

*This is weird:* with  $\Lambda > 0$  then whatever the “stuff” that accounts for  $\Lambda$ , it exerts *negative* pressure! But with  $\Lambda$  in the theory at least neither the energy density or pressure of *ordinary* matter or radiation need to acquire negative values. Whatever it *is*, the introduction of  $\Lambda$  turns the field equations (15) and (16) into

$$e^{-B(r)} (A'/r + 1/r^2) - 1/r^2 + \Lambda = 8\pi G P, \quad (26)$$

$$e^{-B(r)} (B'/r - 1/r^2) + 1/r^2 - \Lambda = 8\pi G \rho, \quad (27)$$

and Eq. (17) remains unchanged;  $\rho$  and  $P$  denote here the energy density and pressure ordinary gravitating matter and radiation. When these are uniform we find

$$e^{-B(r)} = 1 - (\Lambda - 8\pi G P_0) r^2. \quad (28)$$

If we re-define the “Einstein radius”  $a_E$  as

$$a_E^2 \equiv (\Lambda - 8\pi G P_0)^{-1} \quad (29)$$

then  $e^{-B(r)} = 1 - r^2/a_E^2$ , yielding an acceptable metric for a *closed* universe of spherical geometry,

$$d\tau_E^2 = dt^2 - (1 - r^2/a_E^2)^{-1} dr^2 - r^2 d\Omega^2. \quad (30)$$

From Eq. (29) we have a constraint between the pressure and the radius of curvature  $a_E$ , which shows again that if  $\Lambda = 0$ , then ordinary radiation in Einstein’s model universe would have to exert unphysical *negative* pressure. When Eq. (29) is imported into Eq. (27) another constraint on Einstein’s universe emerges:

$$\Lambda = 4\pi G(\rho_0 - P_0). \quad (31)$$

Most of the energy density resides in the ponderable matter of stars, while the pressure comes from starlight and is relatively small. Approximating the latter as zero, Einstein’s model gives a tightly constrained one-parameter system,

$$\Lambda = 1/a_E^2 = 4\pi G \rho_0. \quad (32)$$

At the time Einstein developed his model, the average density of the universe was estimated at about  $1.5 \times 10^{-30}$  g/cm<sup>3</sup>. Einstein was criticized for creating a one-parameter universe.[12] But that was where GR and the phenomenology of the day led him, over “a rather rough and winding road.”

Although Einstein reached for the cosmological constant only after all other options were seemingly exhausted, such *ad hoc* moves were contrary to his style of reasoning from fun-

damental principles. He concluded the 1917 paper with a disclaimer:

*In order to arrive at this consistent view, we admittedly had to introduce an extension of the field equations of gravitation which is not justified by our actual knowledge of gravitation...That term is necessary only for the purpose of making possible a quasi-static distribution of matter, as required by the fact of the small velocities of the stars.*

Einstein had solved the problem at infinity, but he did not like his solution. However, his paper started a discussion that would blossom over the next decade into the genesis of big bang cosmology. In the next installment we will look some more at Einstein’s model, and an alternative to it proposed by his good friend, the Dutch astronomer Wilhelm de Sitter, whose alternative solution to Einstein’s model was both strange and suggestive.

## REFERENCES

- [1] “Elegant Connections in Physics: History of Big Bang Cosmology, Part 1: Cosmic Geography”
- [2] *idem*, “The Principle of Equivalence”
- [3] E. Taylor & J. A. Wheeler, *Spacetime Physics* (Freeman, 1966 & 1992), Ch. 1.
- [4] C. Misner, K. Thorne, & J. A. Wheeler, *Gravitation* (Freeman, 1973), Ch. 1.
- [5] For more practice at this sort of thing, see E. Taylor & J. A. Wheeler, *Black Holes* (Freeman, 2000).
- [6] S. Weinberg, *Gravitation and Cosmology* (Wiley, 1972), p. 7.
- [7] There are 10 independent components, not 16 in four-dimensional spacetime, because the tensors are symmetric.
- [8] The affine connection coefficients are *not* tensor components. Neither is the derivative of a tensor, in general, also a tensor. The non-tensor part of the affine connection cancels the non-tensor part of the derivative, so that the “derivative plus affine connection” is a tensor, and thus may be written in all coordinate systems. Specifically,  $\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\xi\mu} \{ \partial_\nu g_{\sigma\xi} + \partial_\sigma g_{\nu\xi} - \partial_\xi g_{\sigma\nu} \}$ , where  $\partial_\nu$  means  $\partial/\partial x^\nu$ , and the  $g^{\xi\mu}$  are the components of the multiplicative inverse of the matrix of metric tensor components. Also,  $R_{\mu\nu} = \partial_\nu \Gamma^\sigma_{\mu\sigma} - \partial_\sigma \Gamma^\sigma_{\mu\nu} + \Gamma^\sigma_{\xi\nu} \Gamma^\xi_{\mu\sigma} - \Gamma^\sigma_{\xi\sigma} \Gamma^\xi_{\mu\nu}$ .
- [9] The paper can be readily found in the Dover anthology of original papers by Einstein, Lorentz, Weyl, and Minkowski in *The Principle of Relativity*, tr. by W. Perrett & G. B. Jeffrey (Dover, 1952). Einstein’s 1917 cosmology paper was “Cosmological Considerations on the General Theory of Relativity,” pp. 177-188 in the Dover reprint. The article appeared originally as “Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie,” in *Sitzungsberichte der Preussischen Akad. d. Wissenschaften*, 1917.
- [10] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford, 1934; Dover, 1987), Ch. 10.
- [11] If the surface of a sphere in four dimensions is called a “hypersphere,” shouldn’t the surface of a hyperboloid in four dimensions be called a “hyperhyperboloid”?
- [12] Tolman, Ref. 10, p. 463. See also A. Einstein, *The Meaning of Relativity* (Princeton Univ. Press, 5th Ed. 1956), p. 111. Ironically, today’s “standard model” of particle physics is criticized because it has too *many* parameters.

