On the Electrodynamics of Moving Bodies (Part B: Electrodynamics), and its Corollary, $E=mc^2$, by Albert Einstein

ELEGANT CONNECTIONS IN PHYSICS

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This article continues the annotation of one of Einstein's great papers of 1905, "On the Electrodynamics of Moving Bodies."[1] For best results, read these notes with the paper also before you,[2,3] as together we enter the mind of Einstein.[4] Because his famous "$E = mc^2$" paper came about as a logical consequence of the longer electrodynamics paper, I have merged the two papers in this article.

I encourage you to read Einstein's papers, if done alongside this article, as follows:

(2) Read a section of Einstein's paper first.

(b) Try to fill in his omitted steps on your own.

(c) If you get stuck, or when you want to check your work against another student, resume reading this article.

Are you ready? Here we go...

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In Part B of his great paper Einstein turns to the electrodynamics part of "On the Electrodynamics of Moving Bodies."[1] He began this paper, as he began so many of his papers, by drawing our attention to an under-appreciated equivalence, this one found in electric and magnetic forces:

It is well known that Maxwell's electrodynamics—as usually understood at present—when applied to moving bodies, lead to asymmetries that do not seem to be inherent in the phenomena.

Einstein illustrated the "asymmetry" with a magnet and a conducting loop. A magnet gliding through a conducting loop at rest drives a current thanks to Faraday's law. But to an observer riding on the magnet, the current arises in the moving loop because the charges on it move in a magnetic field. In the first case the changing **B** induces an electric field **E**, and the *electric* force $q\mathbf{E}$ acts on particles of charge q. In the second case, the force is identified as $q\mathbf{v}\times\mathbf{B}$ where charge q is carried with velocity **v** past the magnet. Why would the *same result* arise from apparently *different mechanisms*? This equivalence could not be a coincidence, reasoned Einstein.

Since all measurements involve the exchange of information limited by the speed of electromagnetic signals, to resolve the "asymmetry" Einstein first had to get to the source of a *conceptual* confusion. No one had ever asked how, in fact, one actually measures the time interval between separated events, or measures the length of a moving object. This formed Part A of his paper, the "Kinematic Part." Here we follow him, with annotations, through Part B, the electromagnetic part, of "The Electrodynamics of Moving Bodies."

SECTION 6: TRANSFORMATION OF MAXWELL'S EQUATIONS

Einstein begins Section 6 with Maxwell's equations for electric and magnetic fields in vacuum. In modern vector notation and SI units, these "Maxwell-Hertz equations" are

$$\nabla\times \mathbf{E} = -\partial\mathbf{B}/\partial t \quad (1a)$$
$$\nabla\times \mathbf{B} = (1/c^2)\, \partial\mathbf{E}/\partial t \quad (1b)$$
$$\nabla\cdot \mathbf{E} = 0 \quad (1c)$$
$$\nabla\cdot \mathbf{B} = 0 \quad (1d)$$

The curl equations say that a changing electric (magnetic) field produces locally a magnetic (electric) field with a whirlpool. The divergence equations say that, away from point sources, the electric and magnetic fields have streamlines that do not diverge. Let these four Equations (1) refer to Lab Frame coordinates,[5] where the **E** and **B** fields are assumed known. In his paper, Einstein writes out the curl equations for **E** and **B**, component by component, six equations in all. For instance, the x-component of the Ampere-Maxwell law, Eq. (1b) reads

$$(1/c^2)\, \partial E_x/\partial t = \partial B_z/\partial y - \partial B_y/\partial z. \quad (2)$$

All six equations have first derivatives of field components, evaluated with respect to position and time coordinates in the Lab Frame. Einstein transforms these derivatives to the Rocket Frame, through the Lorentz transformation derived in Part A. He leaves out the intermediate details. We provide them here.[5]

The Lorentz transformation says

$$t' = \gamma(t - v_R x/c^2), \quad (3a)$$
$$x' = \gamma(x - v_R t), \quad (3b)$$
$$y' = y, \quad (3c)$$
$$z' = z, \quad (3d)$$

where

$$\gamma \equiv (1 - v_R^2/c^2)^{-½}. \quad (4)$$

From these it follows that[6]

$$\partial_t = \gamma[\partial_{t'} - v_R\, \partial_{x'}] \quad (5a)$$
$$\partial_x = \gamma[\partial_{x'} - (v_R/c^2)\partial_{t'}] \quad (5b)$$
$$\partial_y = \partial_{y'} \quad (5c)$$
$$\partial_z = \partial_{z'}, \quad (5d)$$

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where to avoid an avalanche of partial derivative symbols I use subscript notation $\partial_t \equiv \partial/\partial_t$, $\partial_x \equiv \partial/\partial_x$. With Eq. (5) readily at hand, Eq. (2) transforms into Rocket Frame coordinates to become:

$$\gamma(1/c^2)[\partial_{t'} E_x - v_R\, \partial_{x'} E_x] = \partial_{y'} B_z - \partial_{z'} B_y \,. \tag{6}$$

The "new" term $\partial_{x'} E_x$ also appears in Gauss' law for **E**, which in Lab Frame coordinates may be written

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = 0. \tag{7}$$

With the Lorentz transformation, Eq. (7) becomes

$$g[\partial_{x'} E_x - (v_R/c^2)\partial_{t'} E_x] + \partial_{y'} E_y + \partial_{z'} E_z = 0. \tag{8}$$

From Eq. (8), let us isolate $g\partial_{x'} E_x$:

$$g\partial_{x'} E_x = g\,(v_R/c^2)\partial_{t'} E_x - \partial_{y'} E_y - \partial_{z'} E_z \,. \tag{9}$$

In Eq. (6) replace $g\partial_{x'} E_x$ using Eq. (9), and do some rearranging to turn the latter into

$$(1/c^2)\partial_{t'} E_x = g\, \partial_{y'} [B_z - (v_R/c^2)E_y] - g\, \partial_{z'} [B_y + (v_R/c^2) E_z] \,. \tag{10}$$

This equation is merely Eq. (2), the Ampere-Maxwell law, for the electric and magnetic fields as measured in the Lab Frame, but with the derivatives mapped to the Rocket Frame coordinates via the Lorentz transformation. By virtue of the Principle of Relativity, the empty-space Maxwell's equations in the Rocket Frame must say that

$$(1/c^2)\, \partial \mathbf{E'}/\partial t' = \tilde{N}'\times\mathbf{B'} \tag{11a}$$
$$-\partial \mathbf{B'}/\partial t' = \tilde{N}'\times\mathbf{E'} \tag{11b}$$
$$\tilde{N}'\cdot \mathbf{E'} = 0 \tag{11c}$$
$$\tilde{N}'\cdot \mathbf{B'} = 0 \tag{11d}$$

Therefore, the x'-component of the Maxwell-Ampere equation, Eq. (11a), for instance, says

$$(1/c^2)\partial_{t'} E'_{x'} = \partial_{y'} B'_z - \partial_{z'} B'_y \,. \tag{12}$$

Now Einstein compares Eqs. (10) and (12). The Relativity Principle requires

$$E'_{x'} = y\,(v_R)\, E_x \tag{13a}$$
$$B'_{y'} = y\,(v_R)\, g\, [B_y + (v_R/c^2)E_z] \tag{13b}$$
$$B'_{z'} = y\,(v_R)\, g\, [B_z - (v_R/c^2)E_y] \tag{13c}$$

where $y\,(v_R)$ denotes an overall proportionality factor that may depend on the relative velocity of the two frames. Through similar steps with the **curl B** and **div B** equations, Einstein also finds

$$B'_{x'} = y\,(v_R)\, B_x \tag{14a}$$
$$E'_{y'} = y\,(v_R)\, g\, (E_y - v_R\, B_z) \tag{14b}$$
$$E'_{z'} = y\,(v_R)\, g\, (E_z + v_R\, B_y) \tag{14c}$$

To determine $y\,(v_R)$, Einstein inverts the transformation two ways, then compares them. In the first way, he inverts the equations algebraically; for example $E_x = E'_{x'} / y\,(v_R)$. But one can also derive the transformation by starting with the Rocket Frame Maxwell's equations and transforming the derivatives of **E'** and **B'** into derivatives with respect to the unprimed Lab Frame fields and coordinates. Here we must keep in mind that the Lab Frame moves with velocity $-v_R$ relative to the Rocket Frame. In this way one finds, for example, that $E_x = y\,(-v_R)E'_{x'}$. Comparing the two results gives $y\,(v_R)y\,(-v_R) = 1$. Einstein notes that, "For reasons of symmetry" $y\,(v_R) = y\,(-v_R)$; therefore $y\,(v_R) = 1$.

To sum up Einstein's first "electrodynamics of moving bodies" result: the fields **E'** and **B'** observed by the Rocket Observer are, in terms of the Lab Frame fields **E** and **B**, given as components according to[7]

$$E_{x'} = E_x \tag{15a}$$
$$E_{y'} = g(E_y - v_R\, B_z) \tag{15b}$$
$$E_{z'} = g(E_z + v_R\, B_y) \tag{15c}$$

and

$$B_{x'} = B_x \tag{15d}$$
$$B_{y'} = g(B_y + v_R\, E_z/c^2) \tag{15e}$$
$$B_{z'} = g(B_z - v_R\, E_y/c^2) \tag{15f}$$

Einstein summarizes his results of this section with these remarks:

The above equations can be expressed in words in the following two ways:

*1. If a point charge q moves in an electromagnetic field, in addition to the force q***E** *acting on it there is also the force q***v**×**B** *(Old mode of expression).*

*2. If a point charge moves in an electromagnetic field, the force on it equals the electric force q***E'** *where the* **E'** *is obtained by transforming the field to a coordinate system at rest relative to the charge (New mode of expression).*

Analogous remarks hold for magnetic forces. We can see that in the theory developed here,...electric and magnetic forces do not have an existence independent of the state of motion of the coordinate system.

It is further clear that the asymmetry in the treatment of currents produced by the relative motion of a magnet and a conductor, mentioned in the introduction, disappears.

In the remainder of the paper, Einstein pursues further applications of the relativity of electrodynamics.

Section 7: Doppler Principle and Aberration

Consider a harmonic wave in the electromagnetic field. The Lab Frame observer observes the oscillatory electric field **E** and magnetic field **B**,

$$\mathbf{E} = \mathbf{E}_0 \sin \Phi \tag{16}$$

$$\mathbf{B} = \mathbf{B}_0 \sin \Phi \tag{17}$$

with phase

$$\Phi \equiv w\, t - \mathbf{k}\bullet\mathbf{r}. \tag{18}$$

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Here w and **k** denote the wave's angular frequency and wavenumber vector respectively (**k** is normal to the wave front, and points in the direction of propagation), so that

$$c = w / |\mathbf{k}|, \tag{19}$$

or in terms of the Hertzian frequency f and wavelength l,

$$c = fl \tag{20}$$

where

$$w = 2pf \tag{21}$$

and

$$|\mathbf{k}| = 2p/l\ . \tag{22}$$

Let us figure out the frequency f' that will be measured in the Rocket Frame, if we know the frequency f of the signal as measured in the Lab Frame. The frequency lies buried in the phase. So we start with Φ, given in Lab Frame coordinates, and execute the Lorentz transformation to write the phase in terms of Rocket Frame coordinates. To prepare the way, let's write Φ, Eq. (18), in the form

$$\Phi = w[t - (K_x + L_y + M_z)/c] \tag{23}$$

where K, L, and M are the cosines of the angle that the wavevector **k** makes with respect to the x, y, and z axes, respectively. From the inverse Lorentz transformations in Eqs. (3), we find

$$x = g(x' + v_R t'), \tag{24a}$$

$$y = y' \tag{24b}$$

$$z = z' \tag{24c}$$

$$t = g(t' + v_R x'/c^2) \tag{24d}$$

Substituting these into Eq. (23) gives, after much algebra which requires the consumption of a large pot of coffee,

$$\Phi = w'\ [\ t' - (K'x' + L'y' + M'z')/c] \tag{25}$$

where

$$w' \equiv wg(1 - Kv_R/c), \tag{26}$$

and

$$K' \equiv (K - v_R/c)(1 - Kv_R/c)^{-1} \tag{27a}$$

$$L' \equiv (L/g)(1 - Kv_R/c)^{-1} \tag{27b}$$

$$M' \equiv (M/g\ )(1 - Kv_R/c)^{-1}. \tag{27c}$$

We notice that, numerically, Φ = Φ' thanks to the transitive property of the equality relation. But we have the phase given explicitly as functions of the Lab and Rocket Frame coordinates, by Eqs. (23) and (25) respectively.[8]

As seen in the Lab Frame, let the light ray (perpendicular to plane wave fronts) move at angle j relative to the velocity vector $\mathbf{v}_R$ made by the moving rocket. A sketch of this situation, showing the ray and wave fronts, appears in Fig. 1.

Recall that the light has frequency f as measured in the Lab Frame. So as measured from the Rocket Frame the frequency f' is given by Eq. (26),

$$f' = fg\ [1 - (v_R/c)\cos j\ ]. \tag{28}$$

If $j = 0$, so that the light ray's direction and velocity of the Rocket relative to the Lab are co-linear, then

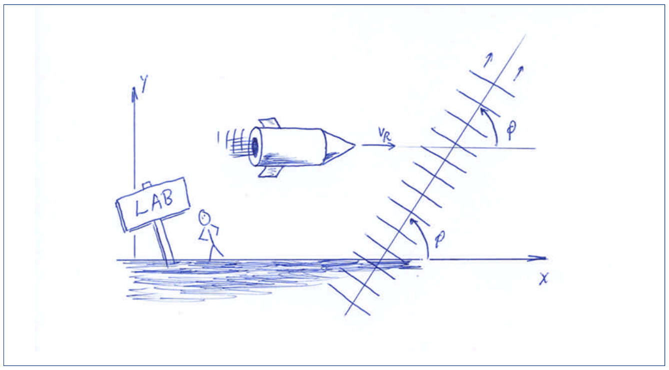

Fig. 1. Motion of a set of wave fronts and their ray. As seen by the Lab Observer, the ray makes the angle j with respect to the Rocket's velocity $\mathbf{v}_R$.

$$f' = f\,[(1 - v_R/c)/(1 + v_R/c)]^{1/2}\ , \tag{29}$$

the Dopper shift as it is commonly presented in textbooks.

By virtue of Eq. (27a), the direction traveled by the light ray, as seen from the Rocket Frame, makes angle j' with respect to the x'-axis, where

$$\cos j' \equiv (\cos j - v_R/c)(1 - \cos j\ v_R/c)^{-1}\ , \tag{30}$$

which Einstein notes is the "law of aberration in its most general form." If the light ray comes in perpendicular to the x-axis, and thus perpendicular to the velocity of the rocket relative to the lab, then it comes across the x'-axis of the Rocket Frame at the angle j' given by

$$\cos j' = -\,v_R/c\ . \tag{31}$$

"Aberration" was well known to astronomers. It's the same effect as having to tip your umbrella when you run fast in a vertical rain. Because the Earth moves relative to the Sun, and the speed of light is finite, one must tip one's telescope in the direction of motion so that a pulse of light entering the top of the moving telescope also hits the mirror at the bottom.

Einstein completes this section by turning his attention to the relativity of the amplitudes of the electromagnetic wave. He needs the amplitudes because the energy carried by radiation goes as the amplitude squared, and electromagnetic energy comes up in Section 8. In Section 7 he is quite terse with the amplitudes, saying,

We still need to find the amplitude of the waves as it appears in the [Rocket frame]. If A and A' denote the amplitudes of [the] electric or magnetic [field] in the [Lab Frame] and [Rocket Frame] respectively, we get

$$A'^{\,2} = A^2 g^2[1 - (v_R/c)\cos j\ ]^2.$$

Perhaps this is obvious by inspection to you, but I had to do some scratch work here. So here goes. Take the amplitude squared of, say, the electric field in the Rocket Frame:

$$E'^{\,2} = E_x'^{\,2} + E_y'^{\,2} + E_z'^{\,2}. \tag{32}$$

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Use now the transformations of Eqs. (15a-c). After a load of algebra, you will come up with

$$E'^{\,2} = E_x^2 + \gamma^2(E_y^2 + E_z^2) - 2v_R\,\gamma^2(E_yB_z - E_z\,B_y) + v_R^2\,\gamma^2(B_z^2 + B_y^2). \quad (33a)$$

To go farther with this, we might recall that the magnitude of a vector, and thus its square, is independent of the spatial coordinates. We can therefore choose the *components* of **E** in the Lab Frame to simplify the right-hand-side of Eq. (33) as follows: consider $E_x = 0 = E_y$, so that **E** has only a z-component. Since **B** points perpendicular to **E**, and **E**×**B** points in the direction the wave travels, **B** lies in the *xy* plane of the Lab Frame. See Fig. 2.

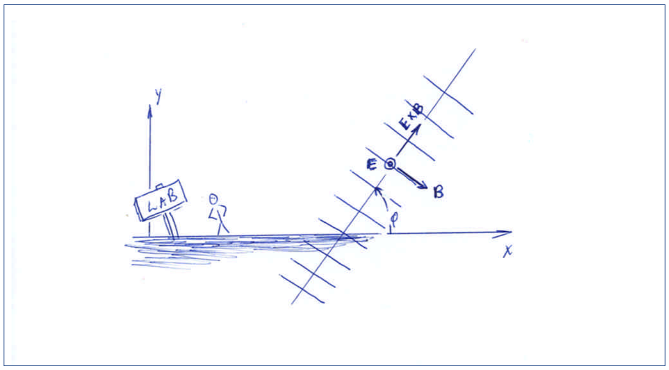

Fig. 2. We consider **E** perpendicular to the plane of the figure and **B** in the plane.

This arrangment turns Eq. (33) into

$$E'^{\,2} = \gamma^2(E^2 - 2v_R\,EB\cos\varphi + v_R^2\,B^2\cos^2\varphi). \quad (33b)$$

Finally we recall that, for radiation fields, |**E**| = c|**B**|, which turns Eq. (33b) into

$$E'^{\,2} = E^2\,\gamma^2[1 - (v_R/c)\cos\varphi]^2 \quad (34)$$

and Einstein's result. For $\varphi = 0$, Eq. (34) gives

$$E'^{\,2} = E^2\,(1 - v_R/c)\,(1 + v_R/c)^{-1}. \quad (35)$$

With Einstein we note that, if the rocket *approaches* the light source at the speed of light (so that $v_R = -c$), then it follows that to the Rocket Observer, "this source would have to appear infinitely intense."

8. Transformation of the Energy of Light Rays, Theory of Radiation Pressure Exerted on Perfect Mirrors

In this section Einstein examines the relativity of electromagnetic energy. The energy *density* of the electric and magnetic fields are given by $\frac{1}{2}\varepsilon_0E^2$ and $\frac{1}{2}B^2/\mu_0$ respectively, where ε_0 and μ_0 denote the permitivity and permeability of empty space. Also, for radiations fields $E = cB$. Therefore the total energy *density* of the electromagnetic field is proportional to E^2. One might suppose that E'^2/E^2 would be the ratio of *energies* of electromagnetic radiation in the Rocket and Lab Frames, but that conclusion presupposes that the energy occupies equal *volumes*. Because of length contraction, "this is not the case."

Therefore, Einstein had to work out the relativity of volumes enclosing a fixed quantity light, a pulse emitted isotopically in all directions from a source at rest in the Lab Frame. In terms of the direction cosines introduced above, the (x, y, z) coordinates of a point on the spherical surface of this light pulse are given by (Kct, Lct, Mct), so the equation of this surface would be

$$(x - Kct)^2 + (y - Lct)^2 + (z - Mct)^2 = R^2. \quad (36)$$

This surface is not traversed by any electromagnetic energy, always enclosing the same complex of radiant energy. The equation of this surface describes a sphere in the Lab Frame. Transforming it to the Rocket Frame with the Lorentz transformation of Eqs. (24), Einstein finds for the equation of the surface an ellipsoid, expressed in Rocket Frame coordinates at time $t' = 0$,

$$(\gamma x' - K\gamma v_R x'/c)^2 + (y' - L\gamma v_R x'/c)^2 + (z' - M\gamma v_R x'/c)^2 = R^2. \quad (37)$$

Einstein then says, "If S denotes the volume of the sphere and S' that of the ellipsoid, then a simple calculation shows that

$$S'/S = \gamma^{-1}\,[1 - (v_R/c)\cos\varphi\,]." \quad (38)$$

Einstein's "simple calculation" requires some scratch work. For an ellipse with symmetry axes of length A, B, and C, we know that the volume is $4\pi ABC/3$. It's not obvious to me (perhaps it is to you) just for the mere looking, how Einstein gets from Eqs. (36) and (37) to Eq. (38). So, like Einstein, we have to think how to show it ourselves. He pays us the respect of assuming we can do it. Here's how I think about it.

The source of light, it will be recalled, lies at rest in the Lab Frame. Let it emit waves of frequency f and wavelength λ, as measured in that frame, and let these waves be emitted isotropically in all directions. Consider the spherical surface in this frame that has a radius of one wavelength. See Fig. 3a. The volume S enclosed by this sphere equals

$$S = 4\pi\lambda^3/3. \quad (39)$$

Now let's look at the "one-wavelength surface" in the Rocket Frame, which, as Einstein has shown, forms an ellipsoid. Consider the Rocket Frame observer for whom (as was seen back in the Lab Frame) the wavefront travels at angle φ with respect to the Rocket's velocity. This Rocket Frame observer measures frequency f' given by Eq. (28), and thus the wavelength

$$\lambda' = \frac{\lambda}{\gamma\,[1 - (v_R/c)\cos\varphi\,]}. \quad (40)$$

As seen by this particular Rocket Frame observer, the axes of the ellipsoid *perpendicular* to the relative motion between observer and source will *not* be Lorentz-contracted; and the wavelength *along* the line joining the observer and source will be contracted to λ'.

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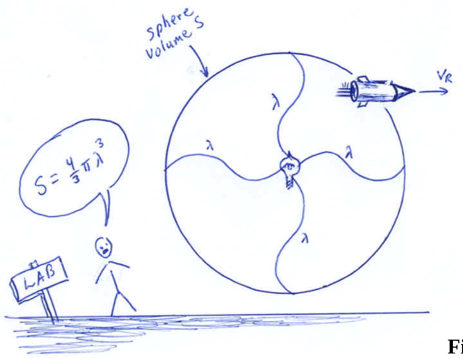

Fig. 3a

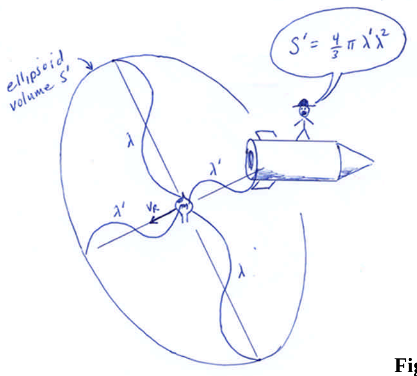

Fig. 3b

Fig. 3. Emission of a wave in all directions. (a) Spherical wave, of radius one wavelength as seen in Lab Frame, where the source is at rest. Note the motion and location of the Rocket Observer relative to the Lab. (b) Same wave seen in the frame of the Rocket Observer.

Therefore

$$S' = 4p l' l^2/3 \,. \qquad (41)$$

With Eq. (40), this leads to Einstein's result, Eq. (38).

Now we can move on with Einstein to a discussion of the relativity of electromagnetic energy. If U' denotes the energy of this light enclosed by the ellipsoid in the Rocket Frame, and U the energy of the *same* light enclosed by the sphere in the Lab Frame, then

$$U'/U = \frac{E'^2 S'}{E' S} = g\,[1 - (v_R/c)\cos j\,], \qquad (42)$$

which simplifies for $j = 0$ to

$$U'/U = [(1 - v_R/c)/(1 + v_R/c)]^{1/2} \,. \qquad (43)$$

This result, Eq. (42), would serve as the starting-point for the world's first derivation of $E = mc^2$. But that was a few weeks in the future.

Here Einstein makes a side comment that echoes the light quantum paper that came a few weeks earlier:

"It is noteworthy that the energy and the frequency of a light complex vary with the observer's state of motion according to the same law."

Next Einstein considers the reflection of radiation. Let the $x' = 0$ plane in the Rocket Frame be a perfectly reflecting mirror from which the plane waves of Section 7 are reflected. Relative to the Lab Frame, the incident radiation has amplitude A, frequency f, and its ray makes angle j with respect to the x-axis. These quantities as measured in the Rocket Frame before reflection will be denoted A', f', and $\cos j'$. They are given in terms of their Lab Frame counterparts by Eqs. (28), (30), and (34). Let these quantities, as measured in the Rocket Frame *after* reflection, be denoted with double primes. According to the laws of reflection, in this frame the post-reflection quantities are related to their pre-reflection values by

$$A'' = A'$$
$$\cos j'' = -\cos j'$$
$$f'' = f'.$$

With Einstein, let us now transform these post-reflection quantities back to the Lab Frame, and denote them, as measured in the Lab Frame, with unprimed symbols and "refl" subscripts (Einstein uses triple primes). By reversing the relative velocity between frames in the reverse transformations of Eqs. (28), (30), and (34) we obtain

$$\begin{aligned} A_{refl} &= A'' g\,[1 + (v_R/c)\cos j''\,] \\ &= A' g\,[1 - (v_R/c)\cos j'\,] \\ &= A\, g^2\,[1 - 2(v_R/c)\cos j + (v_R/c)^2\,]; \end{aligned} \qquad (45a)$$

$$\begin{aligned} \cos j_{refl} &= [\cos j'' + (v_R/c)]/\,[1 + (v_R/c)\cos j''] \\ &= [-\cos j' + (v_R/c)]/\,[1 - (v_R/c)\cos j'\,] \\ &= \frac{2(v_R/c) - [1 + (v_R/c)2\cos j}{1 - 2(v_R/c)\cos j + (v_R/c)^2} \end{aligned} \qquad (45b)$$

and

$$\begin{aligned} f_{refl} &= f''\, g\,[1 + (v_R/c)\cos j''\,] \\ &= f' g\,[1 - (v_R/c)\cos j'\,] \\ &= f\, g^{\,2}\,[1 - 2(v_R/c)\cos j + (v_R/c)^2\,]. \end{aligned} \qquad (45c)$$

Now Einstein uses these results to calculate the light pressure exerted on the mirror. The energy per unit time and per unit area incident on a surface may be written as

Power/area = (energy density)(relative velocity),

where the relative velocity denotes the normal component of that between the signal and mirror. For radiation fields, the electric and magnetic energy densities are equal, so that the energy density of the electromagnetic wave equals $E^2/4pk_e$, where $k_e = 9\times10^9$ Nm^2/C^2 denotes the Coulomb constant. When the light ray, moving at the angle j, overtakes the mirror and reflects, the relative velocity for the incoming ray is

$$c\cos j - v_R,$$

and for the reflected ray the relative velocity is

$$-c\cos j_{refl} + v_R.$$

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Therefore, the difference in power per unit area, ΔI, imparted to the mirror by the incident and reflected waves, equals

$$(1/4pk_e)[E^2(c \cos j - v_R) - E_{refl}^2 (-c \cos j_{refl} + v_R)]. \quad (46)$$

With the help of Eqs. (45b,c) this becomes, after some algebra,

$$\Delta I = 2\, v_R\, (E^2/4pk_e)\, g^2(\cos j - v_R/c)^2\, . \quad (47)$$

This rate of change of energy per unit area equals the rate at which work is done by radiation pressure P. By the definitions of work and pressure, this rate of doing work equals Pv_R. Consequently the light exerts on the moving mirror the pressure

$$P = 2\, (E^2/4pk_e)\, g^2(\cos j - v_R/c)^2\, . \quad (48)$$

For small velocities, to first order in v_R/c, "in agreement with experiment and with other theories, we get"

$$P = 2\, (E^2/4pk_e) \cos^2 j\, . \quad (49)$$

Here Einstein articulates a strategy for studying the electrodynamics of moving bodies:

All problems in the optics of moving bodies can be solved by the method employed here. The essential point is that the electric and magnetic fields of light that is influenced by a moving body are transformed to a coordinate system that is at rest relative to that body. By this means, all problems in the optics of moving bodies are reduced to a series of problems in the optics of bodies at rest.

Einstein will employ this strategy in Section 10, on the dynamics of an electron moving in response to an electromagnetic field.

9. Transformation of the Maxwell-Hertz Equations When Convection Currents Are Taken Into Account

Einstein writes Maxwell's equations for the fields in the vicinity of charged particles present with density r and moving with velocity **v**. In SI units, the curl equations say

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t, \quad (50a)$$
$$\nabla \times \mathbf{B} = (1/c^2)\, \partial \mathbf{E}/\partial t + 4pk_m\, r\mathbf{v}. \quad (50b)$$

where $k_m = k_e/c^2$ denotes the Biot-Savart constant. The divergence equations are

$$\nabla \cdot \mathbf{E} = 4pk_e\, r, \quad (50c)$$
$$\nabla \cdot \mathbf{B} = 0. \quad (50d)$$

If the electric charges are conceived as permanently bound to small, rigid bodies (ions, electrons), then these equations constitute the electromagnetic foundation of Lorentz's electrodynamics and optics of moving bodies.

As in Section 6, here Einstein performs a Lorentz transformation on the space and time coordinates in the derivatives (recall the steps from Eqs. (2) through (12)). The new feature here, not present in the homogeneous case, is the transformation of the charge density r and current density $r\mathbf{v}$. Here Einstein states his procedure and cites the result; let's walk through it in more detail. Start with, say, the x-component of the Amperé-Maxwell law, written for Lab Frame observables:

$$4pk_m\, r v_x + (1/c^2)\, \partial_t E_x = \partial_y B_z - \partial_z B_y\, . \quad (51)$$

Use the Lorentz transformation of Eqs. (5) to shift the derivatives to Rocket Frame coordinates. Eq. (51) becomes

$$4pk_m\, r v_x + (1/c^2)\, \partial_{t'} E_x = \gamma \partial_{y'} [B_z - (v_R/c^2)E_y] - \gamma \partial_{z'} [B_y + (v_R/c^2)E_z]. \quad (52)$$

To write $\partial_{z'} E_z$ in terms of time derivatives we use Gauss' law for **E**, Eq. (50c), which says

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = 4pk_e\, r\, . \quad (53)$$

Writing these derivatives in terms of Rocket Frame coordinates using Eqs. (5), Eq. (53) gives

$$\gamma \partial_{z'} E_z = 4pk_e\, r + \gamma (v_R/c^2)\, \partial_{t'} E_x - \partial_{y'} E_y - \partial_{z'} E_z\, . \quad (54)$$

Replacing the $\gamma\, \partial_{z'} E_z$ in Eq. (52) with the right-hand-side of Eq. (54), noting that $k_e = k_m c^2$, and doing some re-arranging yields

$$4pk_m\, \gamma r\, (v_x - v_R) + (1/c^2)\, \partial_{t'} E_x = \gamma \partial_{y'} [B_z - (v_R/c^2)E_y] - \gamma \partial_{z'} [B_y + (v_R/c^2)E_z]. \quad (55)$$

According to the Principle of Relativity, Eq. (55) must be identical to the Amperé-Maxwell law for Rocket Frame observables,

$$4pk_m\, r'\, \mathbf{v'} + (1/c^2)\, \partial \mathbf{E'}/\partial t' = Ñ' \times \mathbf{B'} \quad (56)$$

Equate the fields of Eqs. (55) and (56) using the previous result [*cf.* Eqs. (15)]; to reconcile the current density terms between the Lab and Rocket Frames we require

$$\gamma r\, (v_x - v_R) = r'\, v_{x'}\, . \quad (57)$$

From the kinematic section, Part A of the paper, Einstein showed the relativity of velocity, which for the x-component says

$$v_{x'} = \frac{v_x - v_R}{1 - v_x v_R/c^2}\, . \quad (58)$$

Therefore, Eq. (57) yields

$$r' = \gamma (1 - v_x v_R/c^2) r\, . \quad (59)$$

*Since [**v**'] is actually the velocity of the electric charges measured in the [Rocket Frame], we have thus shown that, on the basis of our kinematic principles, the electrodynamic foundations of Lorentz's theory of electrodynamics of moving bodies agrees with the principles of relativity.*

Let me also briefly add that the following important proposition can easily be deduced from the equations we have derived: If an electrically charged body moves arbitrarily in space without altering its charge when observed from a coordinate system moving with

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the body, then its charge also remains constant when observed from the [Lab Frame].

10. Dynamics of the (Slowly Accelerated) Electron

In this section, Einstein considers an electrically charged particle with charge e and mass m ("called an 'electron' in what follows"). The electron accelerates so slowly that its radiation can be ignored. He begins by asking us to consider the situation where the electron sits instantaneously at rest in the Lab Frame. It will feel, at least instantaneously, no magnetic force even if a **B** field exists in this frame. Einstein applies Newton's Second Law to determine the electron's motion "during the next instant of time." The force equals $q\mathbf{E}$, so component by component, cause and effect of the electron's motion are related by

$$eE_x = md^2x/dt^2 \quad (60a)$$
$$eE_y = md^2y/dt^2 \quad (60b)$$
$$eE_z = md^2z/dt^2 \; . \quad (60c)$$

Now Einstein imagines the electron to be moving through the Lab Frame. In the "old way" of thinking, the force on the electron equals $q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$. But Einstein uses his strategy described earlier, to examine the electron from *another* frame where it does *not* move at the instant in question. He follows his own advice, made at the end of Section 8. Physics is simple when the particles are (at least instantaneously) at rest! Analyze the problem in *that* frame, and afterwards Lorentz-transform the motion to whatever frame you want! So here in Section 10 Einstein confronts an electron moving in the Lab Frame, by first writing the equations of motion in the Rocket Frame chosen to be the frame where the electron is instantaneously at rest. The results can be transformed back to the Lab Frame afterwards, to tell the observer there the laws of the electron's motion.

To pull this off, Einstein imagines the instant when the electron moves through the Lab Frame at the *same* velocity with which the Rocket flies through the Lab Frame. This means of course, that in the Rocket Frame the electron appears instantaneously at rest. *This important stipulation leads to a subtle technical point in following Einstein's next steps.* To illustrate what I mean, let's go outside Einstein's deliberations and consider the motion of the particle with generic velocity vector (u_x, u_y, u_z) as measured in the Lab Frame, and (u'_x, u'_y, u'_z) in the Rocket Frame (where u_x denotes dx/dt, u'_y denotes dy'/dt', and so forth). As usual, let the relative velocity between the frames be $\mathbf{v}_R$, parallel to the x and x' axes. The subtle point will arise in the existence of three distinct gamma factors, as we shall see. Einstein does not comment on the *conceptual* distinction between them, but makes use of their *numerical* relations in the special case $\mathbf{u} = \mathbf{v}_R$. Note that here in this digression I do not, at first, require $\mathbf{u} = \mathbf{v}_R$; when we impose $\mathbf{u} = \mathbf{v}_R$ then we will have Einstein's result of the first part of Section 10.

With Einstein, we *will* suppose, at a given instant, that $\mathbf{u}' = \mathbf{0}$; the particle experiences for this instant no magnetic force in the Rocket Frame. Physics is simple in this Rocket Frame! In describing the physics of this instant, the Rocket Frame observer writes Newton's Second Law as it applies to the electron:

$$eE'_{x'} = md^2x'/dt'^2 \quad (60a)$$
$$eE'_{y'} = md^2y'/dt'^2 \quad (60b)$$
$$eE'_{z'} = md^2z'/dt'^2 \; . \quad (60c)$$

We now transform the Rocket Frame observables from the Rocket Frame to the Lab Frame, using the Lorentz transformation for the spacetime coordinates (Eqs. 3) and the transformations of the fields from Eqs. (15). Let's start with du'_x/dt', the acceleration in the x'-direction. We recall the relativity of velocity, which follows from Eqs. (3), and for the x-velocities gives

$$u'_x = dx'/dt'$$
$$= (dx - v_R dt)/(dt - v_R dx/c^2)$$
$$= (u_x - v_R)/h \; . \quad (61)$$

For brevity I have defined

$$h \equiv 1 - v_R\, u_x/\, c^2 \; . \quad (62)$$

To calculate the acceleration in the Rocket Frame we must evaluate

$$du'_x/dt' = d[(u_x - v_R)/h]/dt' \; . \quad (63)$$

For any quantity F, notice that

$$dF/dt' \;=\; (dF/dt)(dt'/dt)^{-1}. \quad (64)$$

Here comes the technical point that one can easily overlook in trying to re-create Einstein's derivation. Recall the time dilation formula: any particle carries its own proper time, its "wristwatch time"[5*], denoted ds. For a tick of the wristwatch of duration ds, the clocks of observers in the Lab Frame record the time interval

$$dt = \gamma(u)\, ds \; , \quad (65a)$$

and observers in the Rocket Frame record

$$dt' = \gamma(u')\, ds \; , \quad (65b)$$

where

$$\gamma(u) \equiv (1 - u^2/c^2)^{-½} \quad (66a)$$

and

$$\gamma(u') \equiv (1 - u'^2/c^2)^{-½} \; . \quad (66b)$$

The γ factor we have been using with the relative velocity *between frames* will henceforth be distinguished by the notation γ_R,

$$\gamma_R \equiv (1 - v_R^2/c^2)^{-½} \quad (67)$$

OK, so on the one hand we have

$$dt'/dt = \gamma(u')/\, \gamma(u) \quad (68)$$

while on the other hand, from the Lorentz transformation of Eq. (3a) we also have

$$dt'/dt \;= \gamma_R\, h \; . \quad (69)$$

We now have a statement about the relativity of the gamma-factors:

(continued on next page)

$$\gamma(u')/\gamma(u) = \gamma_R h \,. \quad (70)$$

If (as I did) you don't make this distinction between them in your first run-through of re-creating Einstein's derivation, you will have a gamma cubed in the "wrong" place, and an "extra" h. The use of Eq. (70) clarifies the situation and gives Einstein's result.

Returning to the task at hand, which was to evaluate dF/dt', we see we can write it two ways:

$$dF/dt' = (dF/dt)\,\gamma(u')/\gamma(u) = (dF/dt)(\gamma_R h)^{-1} \,. \quad (71)$$

With this, I originally found

$$du'_x/dt' = (du'_x/dt)\,(\gamma_R h)^{-3} \,. \quad (72)$$

But using Eq. (70), this may also be written

$$du'_x/dt' = (du'_x/dt)\,[\gamma(u)/\gamma(u')]^3 \,. \quad (73)$$

We return now to Einstein's narrative. He said explicitly that, *"Without loss of generality, we may and shall assume that the electron is at the coordinate origin and moves with velocity [v_R] along the x-axis of the [Lab Frame] at the moment with which we are concerned. It is then obvious that at the given moment (t = 0), the electron is at rest relative to [the Rocket Frame]."* This means $\gamma(u') = 1$, $h = 1/\gamma_R^2$, and $\gamma(u) = \gamma_R$. With these results, and using the field transformations of Eq. (15), we find that Eq. (60a), the x'-component of $e\mathbf{E}' = ma'_x$, transforms to the Lab Frame as

$$eE_x = m\gamma^3(u)\, d^2x/dt^2 \,. \quad (74)$$

This was the result as Einstein presented it, using the same symbol for $\gamma(u)$ and γ_R. As we have seen , *in general* they are *not* equal, but in the instant considered by Einstein, they *are* equal.

We find in a similar manner the transformations for the accelerations of the y' and z' components, noting that, at the instant in question, Einstein puts $dy/dt = 0$ and $dz/dt = 0$. We obtain

$$e\,\gamma_R\,(E_y - v_R B_z) = m\,\gamma^2(u)\, d^2y/dt^2 \quad (75a)$$

If we now invoke $\gamma_R = \gamma(u)$, then with Einstein we obtain

$$e\,(E_y - v_R B_z) = m\,\gamma(u)\, d^2y/dt^2 \,. \quad (75b)$$

Similarly, the z' equation transforms to

$$e\,(E_z + v_R B_y) = m\,\gamma(u)\, d^2z/dt^2 \,. \quad (76)$$

We have written physics in the Rocket Frame, where the particle, for an instant, experienced electric force $q\mathbf{E}'$ and zero magnetic force. We then transformed the physics to the Lab Frame, to behold the equations of motion for the electrodynamics of the *moving* body.

Einstein next makes a comment about "longitudinal mass" and "transverse mass." He notices that Newton's Second Law when transformed to the Lab Frame, gives a different coefficient than plain old m on the ma side of $F = ma$. Einstein says we can "preserve the equation

Mass × Acceleration = Force"

if we stipulate a re-definition of mass: for forces in the direction of the velocity, we define the "longitudinal mass,"

$$m_{long} \equiv m\,\gamma^3(u)$$

whereas for forces perpendicular to the particle's velocity we define the "transverse mass,"

$$m_{tran} \equiv m\,\gamma^2(u) \,.$$

"Longitudinal" and "transverse" masses are not concepts often taught today; they came out of pre-relativity discussions of the electrodynamics of moving bodies. Einstein tips his hat to that contemporary discussion by saying "Following the usual approach,..." when mentioning these "masses." Einstein concludes this comment with an insightful caution about definitions:

Of course, with a different definition of force and acceleration we would obtain different values for these masses; this shows that we must proceed very cautiously when comparing various theories of electron motion.

As Einstein's work shows, the gamma factors come from the relativity of space and time between the Lab and Rocket Frames; they have nothing to do with mass itself. It's interesting that he writes the same m for the electron in both the Lab and the Rocket Frames. There's nothing to be gained by attaching gamma factors to the m and calling the result some kind of other mass. Leaving semantics and returning to physics, Einstein continues by noting that his equations of motion do not depend on the particle being literally an electron:

It should be noted that these results about mass are also valid for ponderable material points, because a ponderable material point can be made into an electron (in our sense of the word) by adding to it an arbitrarily small electric charge. [original emphasis]

For his last derivation in this famous paper, Einstein will "now determine the kinetic energy of an electron." Let the electron start from the origin of the Lab Frame, with zero velocity, and move along the x-axis due to an electrostatic force $e\mathbf{E}$ which also points along the x-axis. The work done by this force will be

$$\int eE\,dx \,.$$

Einstein will equate this work to the increase in kinetic energy, using the work-energy theorem. In a step to follow Einstein will consider motion only along the x-axis, and write acceleration du/dt as $u\,du/dx$.

Since the electron is supposed to accelerate slowly, and consequently cannot emit any energy in the form of radiation, the energy taken from the electrostatic field must be equated to the kinetic energy K [Einstein used W] *of the electron. Bearing in mind that [Eq. (74)] holds throughout the entire process of motion, we obtain*

$$K = \int eE\,dx$$
$$= \int m\,\gamma^3(u)\,(u\,du/dx)\,dx$$
$$= m\int_0^v \gamma^3(u)\,u\,du$$

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$$= mc^2 [\gamma(v) - 1]. \quad (77)$$

"Thus, K becomes infinitely large when v = c. As is the case for our previous results, superluminal velocities are not possible."

By reasoning similar to Einstein's statement above, this expression for kinetic energy must hold for ponderable bodies.

He wraps up the paper by summarizing his results:

Let us now enumerate the properties of the electron's motion resulting from the system of equations [Eqs. (74-76)] *that are accessible to experiment.*

First, from Eq. (75b) (the y-component) it is possible to determine the velocity of an electron in crossed **E** and **B** fields, where the forces cancel out, in which case $v = E/B$. Einstein notes, "This relation can be tested experimentally since the velocity of the electron can also be measured directly, e.g., using rapidly oscillating electric and magnetic fields."

Second, from the derivation of the kinetic energy, the change in electric potential $\Delta\phi$ must be related to the electron's velocity by

$$\Delta\phi = (m/e)c^2 [\gamma(v) - 1]. \quad (78)$$

Third, the radius of curvature R of an electron moving through a magnetic field **B** acting perpendicular to the electron's velocity will be given from Eq. (75) by

$$v^2/R = (e/m)\, v\, B_z [1 - v^2/c^2]^{½} \quad (79)$$

giving the predicted radius of curvature

$$R = (mv/eB_z) [1 - v^2/c^2]^{½} \quad (80)$$

These three relations are a complete expression of the laws by which, according to the theory presented here, the electron must move.

In conclusion, let me note that my friend and colleague M. Besso steadfastly stood by me in my work on the problem discussed here, and that I am indebted to him for several valuable suggestions.

"On the Electrodynamics of Moving Bodies" was published in *Annalen der Physik* in June 1905. A few months later, Einstein thought of another consequence of the theory, also with observable consequences: the equivalence of mass and energy. In a letter to Conrad Habicht of the summer of 1905, Einstein wrote[2]

One more consequence of the paper on electrodynamics has also occurred to me. The principle of relativity, in conjunction with Maxwell's equations, requires that mass be a direct measure of the energy contained in a body; light carries mass with it. A noticeable decrease of mass should occur in the case of radium. The argument is amusing and seductive; but for all I know the Lord might be laughing over it and leading me around by the nose.

The $E = mc^2$ paper was published in *Annalen der Physik* in September 1905. It was very short, a kind of afterthought to the long electrodynamics paper earlier that spring. The short note was titled

Does the Inertia of a Body Depend on Its Energy Content?

Annalen der Physik **18**, 639-641 (1905)

The results of an electrodynamic investigation recently published by me in this journal lead to a very interesting conclusion, which will be derived here.

This "interesting conclusion" was $E = mc^2$.

Einstein recalls Section 8 of his "Electrodynamics of Moving Bodies" paper, where the ray of a system of plane light waves carries energy ε relative to the Lab Frame and makes the angle φ with Lab Frame x-axis. The Rocket Frame moves parallel to the Lab x-axis, with velocity v_R. Recalling what for us here is Eq. (42),

$$\varepsilon' = \varepsilon\, \gamma\, [1 - (v_R/c)\cos\varphi] \quad (81)$$

where $\gamma = \gamma_R$, Einstein now lays out the following scenario:

Let a body at rest in the Lab Frame have energy E_0 relative to the Lab Frame, and energy E_0' relative to the Rocket Frame. Of course, Rocket Observers see this body moving with velocity $-v_R$ relative to them.

Suppose this body emits plane light waves with energy ε as measured relative to the Lab Frame. Let half the energy carried by these waves move in the direction that makes angle φ with respect to the x-axis, and the other half of the energy be carried in the opposite direction. Let E_1 and E_1' denote the energy *of the body after* emission of light, as measured in the Lab and Rocket Frame respectively. The body remains at rest with respect to the Lab Frame, and continues moving with its original velocity relative to the Rocket Frame, because the light waves emitted in opposite directions carry opposite momentum.

According to the Principle of Relativity, conservation of energy must be true in both reference frames. Energy conservation gives, in the Lab Frame,

$$E_0 = E_1 + [½\varepsilon + ½\varepsilon], \quad (82)$$

and in the Rocket Frame,

$$\begin{aligned} E_0' &= E_1' + \{½\varepsilon\, \gamma [1 - (v/c)\cos\phi] \\ &\qquad + ½\, \varepsilon\, \gamma [1 + (v/c)\cos\phi]\} \\ &= E_1' + \varepsilon\, \gamma. \end{aligned} \quad (83)$$

By subtraction we have

$$(E_0' - E_0) - (E_1' - E_1) = \varepsilon (\gamma - 1). \quad (84)$$

Einstein says, *"Both differences of the form E'- E occurring in this expression have simple physical meanings."* The term $(E_0' - E_0)$, for instance, compares the energy values of the body *before* emission in the Rocket and Lab Frames. Within an additive constant C, this energy difference will be the body's kinetic energy as measured in the Rocket Frame, before the light was emitted:

$$E_0' - E_0 = K_0' + C \quad (85)$$

where K denotes kinetic energy. Likewise, after emission we may

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write

$$E_1' - E_1 = K_1' + C. \quad (86)$$

since C does not change during the emission of light. Eq. (84) becomes

$$K_0' - K_1' = \varepsilon\,(\gamma - 1). \quad (87)$$

The kinetic energy of the body with respect to the [Rocket Frame] decreases as a result of emission of the light by an amount that is independent of the properties of the body. Furthermore, the difference $K_0' - K_1'$ depends on the velocity in the same way as does the kinetic energy of an electron.

In this last sentence Einstein refers to his derivation, in Section 10 of the *Electrodynamics* paper [and our Eq. (77)], that the kinetic energy of a moving electron is $mc^2\,(\gamma - 1)$.

Recalling that $\gamma = (1 - v^2/c^2)^{-½}$, upon expanding the binomial, to leading order in v/c, we get

$$K_0 - K_1 = ½\,(\varepsilon/c^2)\,v^2. \quad (88)$$

From this equation one immediately concludes:

If a body emits the energy ε in the form of radiation, its mass decreases by ε/c^2. Here is it obviously inessential that the energy taken from the body turns into radiant energy, so we are led to the more general conclusion:

The mass of a body is a measure of its energy content; if the energy changes by E, the mass changes in the same sense by $E/9\times10^{20}$ if the energy is measured in ergs and the mass in grams.

It is not excluded that it will prove possible to test this theory using bodies whose energy content is variable to a high degree (e.g., radium salts).

If the theory agrees with the facts, then radiation carries inertia between emitting and absorbing bodies.

In 1905 the strongly radioactive element radium was an energetic enigma, having been discovered in 1898 by Marie and Pierre Curie. The mechanism for its enormous energy release could not be understood until after the atomic nucleus was discovered by Rutherford in 1911. By comparing the masses of products to reactants in the first known produced nuclear reaction done in 1919,

$${}_2He^4 + {}_7N^{14} \rightarrow {}_8O^{17} + p$$

using alpha particles from a radioactive source, Rutherford dramatically confirmed $E = mc^2$.

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References

[1] Albert Einstein, "Zur Elektrodynamik Bewegter Körper," *Annalen der Physik* **17**, 891-921 (1905).

[2] Einstein quotes in this annotation are from the translation by John Stachel (Ed. and trans.) and Roger Penrose, *Einstein's Miraculous Year: Five Papers that Changed the Face of Physics* (Princeton Univ. Press, 1998). This excellent translation includes a chapter of background and history of each of the five 1905 papers.

[3] An earlier translation of Einstein's 1905 relativity papers is W. Perrett and G.B. Jefferey (trans.), *The Principle of Relativity* (Methuen, 1923; reprinted by Dover, 1952).

[4] In 1912 Einstein wrote a summary of Special Relativity in which some of his 1905 calculations re-appear. This was recently published with translation as *Einstein's 1912 Manuscript on the Special Theory of Relativity* (Braziller, 2003).

[5] I can only assume in this article that the reader is familiar with Part A. Translation dictionary for notation between this annotation and Einstein:

This annotation	**Einstein**
Lab frame*	Rest system K
Rocket frame*	Moving system k
speed of light: c	V
Relative velocity between reference frames:	
v_R	v
$(1 - v_R^2/c^2)^{-½}$: γ	β
Lab coordinates (x, y, z, t)	(x, y, z, t)
Rocket coordinates (x', y', z', t')	(*x*, *h*, *ı* , *t*)
Electric field: (E_x, E_y, E_z)	(X, Y, Z)
Magnetic field: (B_x, B_y, B_z)	(L, M, N)
frequency: f	ν
Direction cosines: (K, L, M)	(a, b, c)
Energy of light: U	E
Fundamental charge: e	ε
Electron mass: m	μ

* The "Lab frame" and "Rocket frame" were introduced by E. Taylor and J. A. Wheeler, *Spacetime Physics* (Freeman, 1966). The Lab frame records events in space and time with unprimed coordinates (t, x, y, z); the Rocket frame employs primed coordinates (t', x', y', z'). The respective axes such as x and x' are parallel; the clocks throughout both frames read zero when the two origins coincide; and the Rocket moves with uniform velocity v_R relative to the Lab parallel to the x axis. In the Rocket frame the Lab moves parallel to the x' axis with velocity $-v_R$.

[6] To see this, consider any $f(x, t)$, where x and t are functions of x' and t', so that

$$f(x, t) = f[x(x', t'), t(x', t')].$$

Then

$$\partial_t f = (\partial_{x'} f)(\partial_t x') + (\partial_{t'} f)(\partial_t t')$$

and similarly for $\partial_x f$. Use the Lorentz transformation to find $\partial_t x'$ and $\partial_t t'$. Einstein writes these derivative relations explicitly in Ref. 4.

[7] In Einstein's versions of these equations, some of his terms and ours here differ by a factor of $1/c$ because he uses different units. Using Einstein's original *approach* we have derived the same results but in SI units. Results similar to ours are displayed in the intermediate electrodynamics text of P. Lorrain, D. Corson, and F. Lorrain, *Fundamentals of Electromagnetic Phenomena* (Freeman, 2000), p. 234; and D. Griffiths, *Introduction to Electrodynamics*, 3rd Ed. (Prentice-Hall, 1999), p. 531.

[8] Einstein's transformation of the phase, showing $\Phi' = \Phi$, anticipates the "invariance of the scalar product in spacetime," with the case

$$\omega' t' - \mathbf{k}' \cdot \mathbf{r}' = \omega' t' - \mathbf{k}' \cdot \mathbf{r}'.$$

The quantum postulates $E = h\nu$ and $p = h/\lambda$ for electromagnetic radiation came shortly after the 1905 electrodynamics paper, but if we use them in the above we find the invariance of the scalar product of the energy-momentum four-vector with the spacetime four-vector, $\langle p'|x'\rangle = \langle p|x\rangle$

[9] Quoted in Stachel, Ref. 2, p. 117.

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