ENGINE EFFICIENCY AND REFRIGERATOR PERFORMANCE

ELEGANT CONNECTIONS IN PHYSICS

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Engine Efficiency

In thermodynamics, an *engine* is a system that turns a portion of heat input into work output. The Second Law of Thermodynamics[1] says that, in a cyclic process, not all of the input heat can be converted entirely to work with no other effect; some of the energy as heat must be dumped to a lower-temperature sink. In this article we will consider "two-temperature engines" that extract all of their heat input Q_H from one heat reservoir at "hot" temperature T_H, and dump heat Q_C to one "cold" reservoir at temperature T_C. According to conservation energy, the heat put in equals the work done by the engine plus the heat coming out.

We use the sign convention that counts heat Q as positive when going *into* a system, and negative when coming out; and work W done *by* the system is positive while work done *on* the system carries a negative sign. Thus, for an engine, $Q_H > 0$, $W > 0$, and $Q_C < 0$. By conservation of energy,

$$Q_H = W + |Q_C| \tag{1}$$

which is illustrated in Fig. 1.

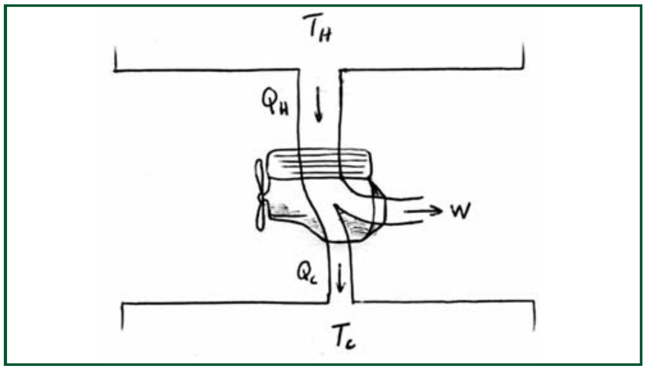

Fig. 1. Energy flow of an engine operating between a "hot" heat reservoir at temperature T_H and a "cold" reservoir at temperature T_C.

Because an engine exists to convert heat input into work output, it makes sense to define the *efficiency* of the engine, denoted e, as the ratio of "what you want" to "what it costs:"

$$e \equiv \frac{\text{what you want}}{\text{what it costs}} = W/Q_H. \tag{2}$$

By the conservation of energy this may also be written

$$e = 1 - |Q_C|/Q_H\ . \tag{3}$$

The Second Law of Thermodynamics says that $|Q_C|$ must be strictly non-zero, so that e must be strictly less than unity,

$$e < 1 \tag{4}$$

because *some* of the energy must be dumped as heat to the low-temperature reservoir. There are no 100 percent efficient engines. If $e < 1$, then *how close* to unity *can* the efficiency be? Carnot's Theorem gives us the answer for a two-temperature engine: one may easily show that any "reversible" (always-in-thermal equilibrium) two-temperature engine, operating between T_H and T_C, will have the efficiency e_o, where

$$e_o = 1 - \beta \tag{5}$$

with

$$\beta \equiv T_C/T_H\ . \tag{6}$$

The efficiency e of an *arbitrary* engine (reversible or not) operating between these same two temperatures will be subject to the inequality

$$e \le e_o \tag{7}$$

where equality holds only in the reversible case. Let

$$\delta \equiv e_o - e\ . \tag{8}$$

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where $\delta \geq 0$; a non-zero δ means the engine has gone out of thermal equilibrium somewhere in its cycle. Thus δ measures the "extra" loss of efficiency *beyond* the requirements of the Second Law of Thermodynamics.

Consider the sum of all the heat per temperature that gets exchanged between a system and its surroundings in a cyclic process. For a system that interacts with the surroundings at only two temperatures, we wish to evaluate

$$\Sigma\, Q/T = (Q_H/T_H) + (Q_C/T_C)\ . \tag{9}$$

Taking into account the signs of the heats, for our engine we have

$$(\Sigma\, Q/T)_{\mathrm{Engine}} = Q_H/T_H - |Q_C|\ /T_C\ . \tag{10}$$

Let's write this sum in terms of one of the heats and one of the temperatures, say Q_H and T_C. From conservation of energy and the definition of engine efficiency we have $|Q_C| = Q_H\,(1 - e)$, and from the expression for the Carnot efficiency we have $1/T_H = (1 - e_o)/T_C$. With these substitutions, and recalling the definition of δ, our sum becomes

$$(\Sigma\, Q/T)_{\mathrm{Engine}} = -\,\delta\, Q_H/T_C \leq 0, \tag{11}$$

an example of the celebrated Clausius Inequality. Equality (the "= 0") occurs only if the engine operates reversibly, always departing negligibly from thermal equilibrium. In that case the cycle proceeds through a reproducible trajectory of equilibrium states—which means that, in the real world, it must proceed with infinite slowness. Since real thermodynamic process occur in finite time, they are typically *out* of equilibrium at least somewhere in the cycle, which shows up as a non-zero value for δ. Thus for a *real* engine with nonzero δ, because the sum of all the Q/T is negative, that means more Q/T comes *out* of the engine than goes into it. This in turn means that the surrounding environment *gains* more Q/T than it *loses*. This is precisely the entropy increase ΔS of the surroundings:

$$\Delta S_{\mathrm{surroundings}} = +\,\delta Q_H/T_C\ . \tag{12}$$

This is also the entropy increase of the universe, $\Delta S_{\mathrm{universe}}$, because the universe can be partitioned into our engine and the surroundings, and for one cycle of the engine's operation,

$$\Delta S_{\mathrm{universe}} = \Delta S_{\mathrm{engine}} + \Delta S_{\mathrm{surroundings}}\ . \tag{13}$$

But $\Delta S_{\mathrm{engine}} = 0$ because S is a state variable, and thus ΔS compares the entropy of the final state to the initial, and in one cycle of operation the engine returns to its original state (whether or not it goes out of equilibrium between those states). But such a result does not hold for the surrounding environment: as a heat reservoir of infinite heat capacity its Q/T has increased irreversibly, so that

$$\Delta S_{\mathrm{universe}} = +\,\delta Q_H/T_C \geq\ 0\ . \tag{14}$$

This statement expresses the Second Law of Thermodynamics in the language of the engine inefficiency factor δ. With a non-zero δ, in one cycle of the engine's operation it dumps *more* heat and does *less* work than required by the Second Law only; and in so doing, the entropy of the universe increases. Indeed, any *spontaneous* process requires an out-of-equilibrium situation, with an increase in the entropy of the universe; a process that never departs from equilibrium will not do anything spontaneously.

EXAMPLE 1: Suppose an engine takes in all of its heat at the temperature T_H = 1200K and all the heat that it exhausts to the surroundings gets rejected at T_C = 300K. The Carnot efficiency—the maximum efficiency possible in principle, limited only by the Second Law of Thermodynamics—is $e_o = 1 - 300\mathrm{K}/1200\mathrm{K} = 3/4$.

Let the engine extract Q_H = 1000 J of energy per cycle from the hot reservoir. Then a Carnot engine operating between these temperatures could perform work $W_o = e_oQ_H$ = 750 J, and reject heat $|Q_{C0}|$ = 250 J to the cold reservoir. However, a real engine will not be as efficient as the Carnot ideal engine. Suppose a real engine operating between these heat reservoirs has an actual efficiency of ¼, so that δ = ½. When extracting 1000 J of energy per cycle from the hot reservoir, this real engine performs work $W = eQ_H$ = 250 J, and rejects 750 J to the cold reservoir. This means that, compared to the Carnot engine, the real engine performs 500 J *less work* per cycle than it *could* have, and dumped 500 J *more heat* to the cold reservoir *than was required* by the Second Law. Therefore, while the sum of all the Q/T was zero for the Carnot engine, the sum for the real engine was

$$1000\mathrm{J}/1200\mathrm{K} + (-750\mathrm{J})/300\mathrm{K} = -5/3\ \mathrm{J/K}\ ,$$

the same number as $-\delta Q_H/T_C = -(1/2)(1000\mathrm{J})/300\mathrm{K}$. In one cycle of the engine's operation, the entropy of the surroundings (and thus the universe) has increased by 5/3 J/K.

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From the perspective of thermodynamics, a *refrigerator* is an engine run in reverse. Let's examine the Second Law in terms of refrigerators.

REFRIGERATOR PERFORMANCE

Thermodynamically, a refrigerator does the inverse procedure of an engine. An engine takes in heat flowing from hot to cold and converts a fraction of this energy into work done *by* the engine; conversely, work done *on* a refrigerator pumps heat from cold to hot, as illustrated in Fig. 2.

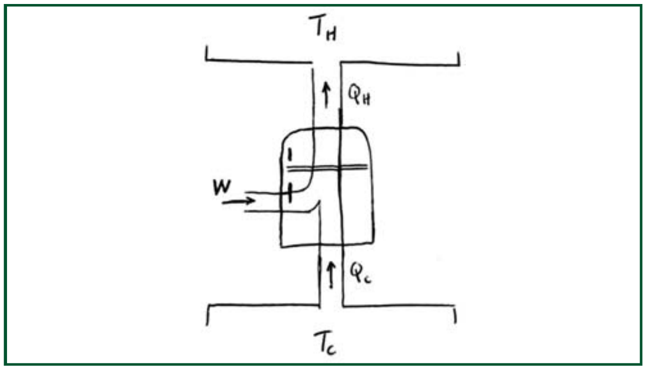

Fig. 2. Refrigerator operation. Work W pumps heat Q_C from the cold reservoir, and energy $Q_C + W$ is pumped to the hot reservoir.

Conservation of energy applied to the refrigerator says

$$|W| + Q_C = |Q_H| \qquad (15)$$

where we have taken note of the fact that, for a refrigerator, $Q_C > 0$, $W < 0$, and $Q_H < 0$.

As we did for the engine, it makes sense to define a "refrigerator efficiency" as "what you want" divided by "what it costs." What we want is Q_C pumped from cold to hot; and that costs some work. Refrigerator efficiency is more commonly called the "coefficient of performance" κ, which by our definition equals

$$\kappa \equiv Q_C/|W| \qquad (16)$$

and by conservation of energy can also be written

$$\kappa = |Q_H|/|W| - 1. \qquad (17)$$

Whether or not the device being used as a refrigerator *could* be run backwards to make an engine, from the definition of an engine's efficiency, Eq. (2), we see that the refrigerator *corresponds to* the efficiency e of a hypothetical engine that *could* operate between the same heat reservoirs as the refrigerator, so that

$$\kappa = 1/e - 1. \qquad (18)$$

This relation *defines* the efficiency e of the "engine equivalent" for the refrigerator operating between two given heat reservoirs. It will be noticed that κ and e are inversely proportional because the work done by the engine is "what we want," but the work put into the refrigerator is "what it costs." The work sits in the numerator for e, but in the denominator for κ.

Because $0 \le e < 1$, it follows that the value of κ will be found on the open interval $(0,\infty)$. For a given engine and refrigerator that work with the same absolute values of Q_H and W, a graph of κ vs. e is shown in Fig. 3. Let's see if we can develop some intuition about numerical values of κ and e. To begin, notice that as an engine becomes more and more efficient at converting heat into work, so that $e \to 1$, then $\kappa \to 0$. What does that mean? As an *engine*, e approaching 1 means that $Q_H \approx W$; very little $|Q_C|$ is dumped to the cold reservoir. Run as a refrigerator, the same system has $\kappa \approx 0$, which means that essentially *no* heat gets pumped from cold to hot. An exceptionally efficient engine makes a very poorly performing refrigerator!

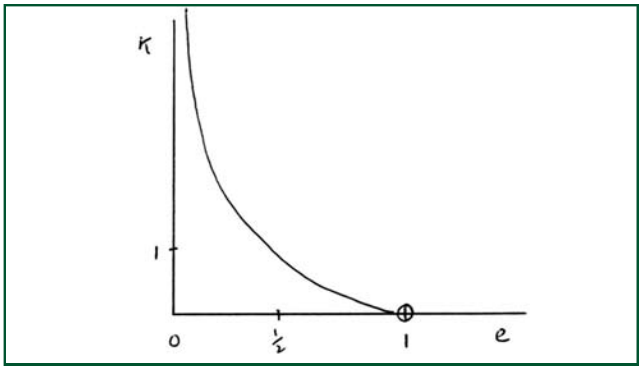

Fig. 3. Refrigerator performance κ as a function of the efficiency e of the corresponding engine.

At the other extreme, as $e \to 0$, the system forms a poor engine; almost all of the heat Q_H put into it gets dumped as heat Q_C; hardly any work gets done. But inversely, the reversed system forms an excellent refrigerator, for with hardly any work input, the heat Q_C gets pumped from cold to hot.

From Eq. (18), $\kappa e = 1 - e = 1 - (e_o - \delta)$, so that

$$\kappa e = \beta + \delta. \qquad (19)$$

If $\delta = 0$ then

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$$\kappa_o \, e_o = \beta \qquad (20)$$

where

$$\kappa_o = 1/e_o - 1 \qquad (21)$$

defines a "Carnot coefficient of performance," a *minimum* coefficient of performance because e_o is the *maximum* engine efficiency attainable in principle for an engine operating between the two temperatures, limited *only* by the Second Law of Thermodynamics. It will be noticed that, from Eqs. (5) and (21) that

$$\kappa_o = (\beta^{-1} - 1)^{-1} = T_C/(T_H - T_C) \qquad (22)$$

Just as a real engine has nonzero δ, likewise the coefficient of performance of a real refigerator differs from the Carnot-equivalent coefficient by some amount $\varepsilon \geq 0$, where

$$\kappa \equiv \kappa_o + \varepsilon \, . \qquad (23)$$

We can write ε in terms of δ. From Eqs. (23) and (18),

$$1/e = 1/e_o + \varepsilon \qquad (24)$$

and with Eq. (8) this becomes

$$\varepsilon = \delta/(ee_o) \, . \qquad (25)$$

If $\delta << e_o$ then to first order in δ,

$$\varepsilon \approx \delta/e_o^2 \, . \qquad (26)$$

However, δ typically is *not* tiny compared to e_o for real engines.

EXAMPLE 2: Let's return to the same heat reservoirs and their temperatures for the engine in our example above, for which $\delta = ½$, and consider the corresponding refrigerator that operates with the same energy budget and two temperatures. A *Carnot* engine run in reverse to make a Carnot refrigerator would have a coefficient of performance $\kappa_o = 1/e_o - 1 = 1/3$, and the refrigerator corresponding to the *real* engine would have the performance coefficient $\kappa = 1/e - 1 = 3$. These parameters, $\delta = ½$ for the real engine, and $\varepsilon = \delta/(ee_o) = 8/3$ for the real refrigerator, measure their departure from the Carnot ideal. We cannot *predict* from thermodynamics alone the values of δ and ε; only the ideal e_o and κ_o are determined from the thermodynamic environment in which the engine or refrigerator find themselves, and then only if all the heat exchanges occur at just two temperatures. Contributions to the values of δ and ε will come from device-dependent factors such as friction and other "extra" heat losses that go *beyond* the requirements of the Second Law. In addition, some of the work put out by the engine, and some of the work put into the refrigerator, is consumed in operating the machine itself.[2]

Let us turn now to a calculation that will enable us to make a statement about entropy, via the sum of all the Q/T exchanged between the refrigerator and its environment in one cycle of the refrigerator's operation. Taking into account the signs of the heats, we have

$$(\Sigma\, Q/T)_{\text{Refrigerator}} = -|Q_H|/T_H + Q_C/T_C. \qquad (27)$$

We might expect, from symmetry, that this sum for the refrigerator reduces to the same factors as its corresponding engine, but with the opposite overall sign because a refrigerator is an engine "run in reverse." We thereby anticipate that

$$(\Sigma\, Q/T)_{\text{Refrigerator}} = +\delta |Q_H|/T_C. \qquad (28)$$

This indeed turns out to be correct, but it forms a useful exercise to see how this answer works itself out in terms of κ. From conservation of energy and the definition of the performance coefficient we have $Q_C = |Q_H|\kappa/(1+\kappa) = \kappa e |Q_H|$. Using also Eq. (6) for β, the sum of the Q/T for the refrigerator becomes

$$(\Sigma\, Q/T)_{\text{Refrigerator}} = |Q_H| \, (\kappa e - \kappa_o e_o)/T_C. \qquad (29)$$

Using $\kappa e = \beta + \delta$ and $\kappa_o e_o = \beta$ yields the result we anticipated, Eq. (28).

Since an engine works *with* a temperature gradient to produce work from heat, and a refrigerator takes work and pumps heat *opposite* the temperature gradient, let's consider the thermodynamics of an engine and a refrigerator working cooperatively, where the work produced by the engine drives the refrigerator.

ENGINE AND REFRIGERATOR WORKING TOGETHER

Let an engine take heat Q_{HE} from a hot reservoir at temperature T_{HE}, perform work W_E, and dump heat Q_{CE} to a

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cold reservoir at temperature T_{CE}. Likewise, let work $|W_R|$ be done on a refrigerator, which pumps heat Q_{CR} from a heat reservoir at cold temperature T_{CR}, with heat $|Q_{HR}|$ deposited to a hot reservoir at temperature T_{HR}. For the engine and refrigerator going through one cycle together, their heat per temperature sum becomes

$$(\Sigma\, Q/T\,)_{E\&R} = \delta_R\, |Q_{HR}|/T_{CR} - \delta_E\, Q_{HE}/T_{CE} \qquad (30)$$

where δ_E denotes the departure from Carnot efficiency for the engine, and δ_R the corresponding quantity for the engine-equivalent of the refrigerator; the latter may be defined in terms of ε.

Let's consider the situation where the work output of the engine matches the work requirement of the refrigerator: $W_E = |W_R| \equiv W$ (see Fig. 4).

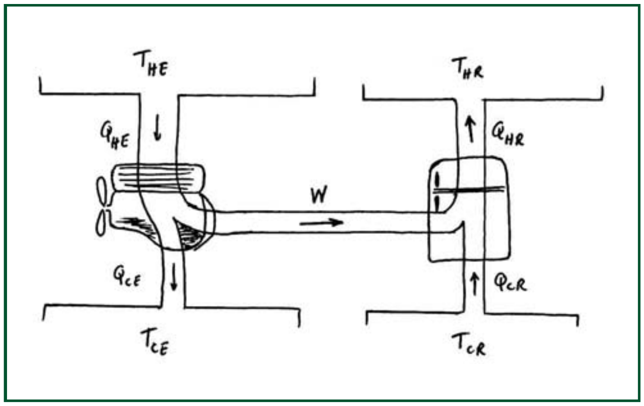

Fig. 4. Tandem engine-refrigerator system, where the work output of the engine matches the work requirement of the refrigerator.

In each cycle of operation the engine must extract heat $Q_{HE} = W/e_E$, where e_E denotes the engine's efficiency, and the refrigerator will reject heat $|Q_{HR}| = |W|(\kappa+1)$ which we write in terms of the equivalent engine efficiency as $|W|/e_R$. Therefore, in terms of inefficiency factors and work, we have

$$(\Sigma\, Q/T\,)_{E\&R} = |W|\,[\delta_R\,/e_R T_{CR} - \delta_E\,/e_E T_{CE}]. \qquad (31)$$

This expression may be alternatively written in terms of temperatures and the heats exchanged with the hot reservoirs. Recall that $e_o - e = \delta$, $eQ_H = W$, and $e_o Q_H = W_o$ where W_o denotes the work that would be done by a Carnot engine operating between the same two temperatures as the actual engine. Also, recalling $e_o = 1-\beta$, Eq. (31) becomes

$$(\Sigma\, Q/T\,)_{E\&R} = (|Q_{HR}|(1-\beta_R)/T_{CR} - Q_{HE}(1-\beta_E)/T_{CE} - |W|(1/T_{CR} - 1/T_{CE}). \qquad (32)$$

Remembering that $\beta = T_C/T_H$, this becomes

$$(\Sigma\, Q/T\,)_{E\&R} = |Q_{HR}|\tau_R - Q_{HE}\tau_E - |W|\frac{(T_{CR} - T_{CE})}{T_{CR}T_{CE}} \qquad (33)$$

where we denote

$$\tau \equiv (T_H - T_C)/T_H T_C. \qquad (34)$$

Notice that τ will be strictly positive if a temperature gradient exists.

For the entropy change of the universe in one cycle of the engine-refrigerator tandem system's operation,

$$\Delta S_{universe} = |W|\,[\delta_E\,/e_E T_{CE} - \delta_R\,/e_R T_{CR}\,] \qquad (35)$$

or alternatively

$$\Delta S_{universe} = -|Q_{HR}|\tau_R + Q_{HE}\,\tau_E + |W|\frac{(T_{CR} - T_{CE})}{T_{CR}T_{CE}}\,. \qquad (36)$$

According to the Second Law of Thermodynamics, $\Delta S_{universe}$ cannot decrease, so that for any process,

$$\Delta S_{universe} \geq 0, \qquad (37)$$

the equality holding only for reversible (always in equilibrium) processes. The strict inequality describes the criteria that the Second Law imposes on spontaneous processes. For our tandem engine-refrigerator system to operate, this requires that

$$|W|\,[\delta_E\,/e_E T_{CE} - \delta_R\,/e_R T_{CR}] > 0 \qquad (38)$$

or, equivalently,

$$-|Q_{HR}|\tau_R + Q_{HE}\,\tau_E + |W|\frac{(T_{CR} - T_{CE})}{T_{CR}T_{CE}} > 0. \qquad (39)$$

A special case of equal cold temperatures, $T_{CE} = T_{CR}$, gives some insight. Then the spontaneous, out-of-equilibrium Inequality (39) collapses to

$$Q_{HE}\,\tau_E - |Q_{HR}|\tau_R > 0. \qquad (40)$$

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Two special cases with $T_{CE} = T_{CR}$ follow.

(1) If the hot temperatures are also equal, $T_{HE} = T_{HR}$ so that $\tau_E = \tau_R$, then $Q_{HE} > |Q_{HR}|$, or, by conservation of energy, $|Q_{CE}| > Q_{CR}$. This says that whenever the engine and refrigerator share the same heat reservoirs, and thus operate between the same two temperatures, then for the engine to drive the refrigerator, the engine must consume more heat from its hot reservoir than the refrigerator returns to it; equivalently, the engine must dump more heat to the cold reservoir than the refrigerator pumps from it. This forms, in essence, a heat conduction process: heat has flowed from hot to cold, through the engine and refrigerator.

(2) If $T_{CE} = T_{CR}$ but we impose $Q_{HE} = |Q_{HR}|$, so that the heat returned by the refrigerator to its hot reservoir equals the heat withdrawn from its hot reservoir by the engine, then according to Inequality (39),

$$\tau_E > \tau_R \tag{41}$$

or

$$T_{HE} > T_{HR} \,. \tag{42}$$

The engine must withdraw heat from a reservoir higher in temperature than that of the hot reservoir to which the refrigerator returns heat. Again, this process describes heat conduction, where heat flows from “hot” to “less hot” spontaneously, through the engine and refrigerator.

Let’s consider now an example that does not assume $T_{CE} = T_{CR}$.

EXAMPLE 3: Let a gasoline-powered engine[3] drive a generator which runs a refrigerator. Let the work output of the engine equal the work input required by the refrigerator. When ignited with a spark under compression, gasoline burns at about 530K = T_{HE}. Suppose the engine runs outdoors where T_{CE} = 70°F ≈ 300K. The Carnot efficiency of an engine operating between these temperatures would be $e_{oE} = 0.434$, but suppose for the real engine that $\delta_E = 0.2$, giving it an actual efficiency of $e_E = 0.23$. Let the engine use 1000 J of energy per cycle. Thus it cranks out 230 J of work per cycle. We assume the refrigerator uses 230 J of work in each cycle of its operation.

Let the refrigerator maintain a chamber that keeps your salami frozen at the freezing point of water, so that T_{CR} = 273K. If the refrigerator also sits on your back porch where it must pump heat to the ambient air, then T_{HR} = 300K. For a Carnot refrigerator these temperatures give $\kappa_o = 10.11$, and a Carnot engine operating between them has $e_o = 0.090$. Suppose (as in Example 2) that $\varepsilon = 8/3$ for the real refrigerator. Then $\kappa = 12.78$, and the “engine equivalent” efficiency for this refrigerator follows from $e = 1/(\kappa+1) = 0.073$. Thus $\delta_R = 0.090 - 0.073 = 0.017$. Now we have everything we need to see if it’s thermodynamically possible for this tandem engine-refrigerator system to run, by evaluating the terms in the left-hand side on Inequality (38) to see if, indeed, the inequality is satisfied. We find that

$$|W| \left[\delta_E / e_E T_{CE} - \delta_R / e_R T_{CR}\right]$$

$$= (230\text{ J})[(0.2)(0.23)^{-1}(300\text{K})^{-1} - (0.017)(0.073)^{-1}(273\text{K})^{-1}] = +0.47\text{ J/K}$$

which is positive; the laws of thermodynamics allow this particular engine to run this particular refrigerator. Each cycle of the tandem system’s operation dumps 0.47 J/K of heat per temperature to the surroundings; in other words the entropy of the universe increases by this amount in each cycle.

ACKNOWLEDGMENTS

Thanks to Don Lemons and Daryl Cox for helpful suggestions.

NOTES

[1] “Elegant Connections in Physics: The Second Law of Thermodynamics and Non-Conservation of Energy,” *SPS Newsletter*, June 1998, pp. 9-13.

[2] Besides efficiency losses due to the Second Law of Thermodynamics, for $\delta > 0$ contributions there are combustion losses, pumping and friction losses, and transmission losses.

[3] Strictly speaking, a gasoline engine takes in and rejects heat across a *range* of temperatures; it’s not a *two*-temperature engine. The gasoline engine is best idealized not by the Carnot cycle but by the Otto cycle. However, I use this example because such an engine/generator/refrigerator arrangement is something we see in everyday life. Because I have restricted this article to a two-temperature engine, what I am really doing in this example is to consider a two-temperature engine where one of these temperatures is that of burning gasoline.

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