A Brief Primer on Gravity Waves, or How to Catch a Wave in Spacetime

Feature

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Anyone with access to a computer can assist in putting Einstein's General Theory of Relativity (GTR) to an important test. GTR predicts that an accelerated mass produces waves in the gravitational field that propagate at the speed of light. Because of the inherent weakness of gravitational radiation, until recently gravity waves could only be inferred *indirectly*, from astrophysical phenomena such as binary star systems losing energy at rates consistent with alleged gravitational radiation. When gravity waves can be detected routinely and *directly*, we will have a new kind of observatory for studying the universe.

The new Laser Interferometer Gravitational-Wave Observatory (LIGO) allows you to help analyze the data collected by this magnificent gravitational wave antenna. To registered users, LIGO downloads batches of data, to be sifted and analyzed when your computer goes into screen-saver mode. At the end of this article you can learn how to become part of the LIGO team.

What physics principles make LIGO possible? In Newtonian gravitation theory, a "source particle" sets up in the space around it a gravitational field. It can be described in the language of the vector field **g** and the corresponding scalar field or potential, ϕ. For an array of source particles of mass increment dm, the potential, as a function of field point **r**, is given by the sum

$$\phi(\mathbf{r}) = -G \int dm/|\mathbf{r} - \mathbf{r}'|$$

where $\mathbf{g} = -\nabla\phi$, $G = 6.67\times10^{-11}$ Nm^2/kg^2 denotes Newton's gravitational constant, and **r**' denotes a source particle's location.

Time as a variable is conspicuous by its absence from Newtonian gravity theory. We do not talk about $\phi(\mathbf{r},t)$. When Newton's apple drops from the tree and changes the Earth's mass distribution, in a Newtonian universe the moon far overhead *instantaneously* feels the change in the Earth's gravitational field. In contrast, when an electron jiggles in an antenna, the change in the electromagnetic field propagates as a wave moving at the speed of light. In a vacuum, this finite speed equals $c = 3\times10^8$ m/s. According to the Special Theory of Relativity (STR, 1905), no signal travels faster than c. Indeed, in STR the speed of light equals c in *all* inertial reference frames, regardless of the relative motion between source and observer. The speed of light is an "invariant." Newtonian gravitation theory collides spectacularly with STR.

In Newtonian theory, space is Euclidian, so an increment of distance ds is measured, in terms of coordinates (x, y, z), by the distance formula

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 .$$

Newtonian time is measured, independent of space, in increments of dt. Distances and time intervals between events are deemed *separately* invariant among reference frames in Newtonian relativity. But STR merges space and time together into a four-dimensional continuum. The "spacetime distance" $d\sigma$ between a pair of nearby events is given by the "Spacetime Interval,"

$$(d\sigma)^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 .$$

Because the speed of light is invariant, distances and times are not separately invariant in STR; however, the Spacetime Interval *is* invariant. We may express the Spacetime Interval in terms of vector components $(cdt, dx, dy, dz) \equiv (dx^0, dx^1, dx^2, dx^3)$ and a matrix η (the "metric tensor of STR") whose components are

$$\eta_{00} = -1,\ \eta_{11} = \eta_{22} = \eta_{33} = 1,$$

and all $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$. The "Spacetime Interval" can be denoted by the summed matrix components

$$-(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2 \equiv \eta_{\mu\nu}\, dx^\mu\, dx^\nu$$

By 1915 Einstein had generalized STR to include physics in accelerated reference frames. GTR states as a postulate the "principle of the equivalence of gravitational and inertial mass." As Galileo showed, all objects fall through the same locality of spacetime with identical acceleration. The spacecraft and the apple turned loose inside it fall together. In Newtonian gravity, this is seen as a deep mystery about *mass*: why should mass *as inertia* equal mass as *gravitational charge*? But in GTR this equivalence suggests an insight about *geometry*: the spaceship and the apple fall together because they are sliding over the same *curved spacetime*. The nearby Earth tells spacetime how to curve. The primary language of gravitation is not "force," but the "curvature of spacetime."

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When an apple falls, a supernova explodes, or two neutron stars orbit violently about their center of mass, according to GTR waves in the gravitational field propagate from them, and sweep outward through spacetime. These changes in the gravitational field will show as a perturbation in the metric tensor shifted from its STR value:

$$\eta_{\mu\nu} \to g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$$

where $g_{\mu\nu} = g_{\nu\mu}$.

Wave solutions for the electric field **E** and the magnetic field **B** can be linearly polarized: the electric wave, for example, can propagate in the *z* direction, but the **E** vector itself can be made to point only parallel to the *x*-axis *or* the *y*-axis. These two independent polarization states are related by a rotation through 90º. Similarly, the wave solutions of Einstein's GTR field equations have two independent polarization states, here denoted Type (1) and Type (2). The polarization states are not given by the orientation of *vectors*, but by *matrices*, because $g_{\mu\nu}$ is a matrix. Thus a gravity wave receiving antenna like LIGO must be a two-dimensional array; it cannot be a one-dimensional wire like a radio antenna.

Let the gravitational wave of frequency ω propagate along the *z*-axis. Frequencies of gravitational waves formed by the collapse of stars to form black holes, or the detonation of a supernova and production of a neutron star, should be in the range of 10^2 Hz to 10^5 Hz. Let this wave arrive at a gravitational receiving antenna consisting of masses arrayed in the *xy* plane. In the two independent polarization states, the waves take the possible forms

$$h_{\mu\nu}^{(1)} = A\ \hat{I}_{\mu\nu}^{(1)} \cos(\omega t)$$

and

$$h_{\mu\nu}^{(2)} = A\ \hat{I}_{\mu\nu}^{(2)} \cos(\omega t)$$

where *A* denotes a generic amplitude. The polarization matrices have 0 for all elements except two, and those occur on the rows and columns for the *x* and *y* coordinates (viz., there are non-zero elements for two of the four entries carrying indices μν = 11, 12, 21, and 22). Ignoring the rows and column that contain only zeroes, we write the 2×2 non-zero subsection of the 4×4 polarization matrices as

$$\hat{I}_{\mu\nu}^{(1)} = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$

and

$$\hat{I}_{\mu\nu}^{(2)} = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$$

Notice that the Type (1) polarization matrix can be obtained by rotating the Type (2) matrix about the *z*-axis by 45º.

Consider a pair of receiver antenna masses in the *xy* plane. In curved spacetime, the *spatial distance* ΔL between them is *not* the Euclidian value $[(\Delta x)^2 + (\Delta y)^2]^{1/2}$, but the spatial part of the Spacetime Interval,

$$(\Delta L)^2 = g_{11}\,(\Delta x)^2 + 2g_{12}\,(\Delta x \Delta y) + g_{22}\,(\Delta y)^2$$

where

$$g_{11} = 1 + h_{11}$$
$$g_{12} = 0 + h_{12} = g_{21}$$
$$g_{22} = 1 + h_{22}\ .$$

Let a Type (2) wave pass over the *xy* plane, and consider the effect on two masses located on the *x*-axis, one at $x = a$, the other at $x = -a$. What happens to the distance between the two masses? We evaluate

$$(\Delta L)^2 = g_{11}\,(\Delta x)^2$$
$$= (1 + h_{11})\,(2a)^2\ .$$

We find from $\hat{I}_{\mu\nu}^{(2)}$ that

$$h_{11}^{(2)} = A\ \hat{I}_{11}^{(2)} \cos(\omega t) = A \cos(\omega t)$$

so that

$$\Delta L = [1 + A\cos(\omega t)]^{1/2}\,(2a)\ .$$

Since gravity fields are weak,

$$\Delta L \approx [1 + \tfrac{1}{2}A\cos(\omega t)]\,(2a)\ .$$

It is crucial to notice the distinction between *coordinate difference* $2a$ and the *distance* ΔL.

For two masses located instead on the *y*-axis at $y = \pm a$, we find the distance between them, $\Delta L'$, to be

$$\Delta L' \approx [1 - \tfrac{1}{2}A\cos(\omega t)]\,(2a)\ .$$

The oscillations of the masses on the *x*-axis are half a cycle out of phase with the oscillations of the masses on the *y*-axis. If our gravity wave antenna has masses on *both* the *x* and *y* axes, and the Type (2) wave moves in the *z* direction, then the masses oscillate as shown in Fig. 1.

Let a beam of laser light be split at the origin into two beams, and let them be sent forth, one along the *x*-axis, the other along the *y*-axis. Both beams are reflected, one at $x = a$, the other at $y = a$. Upon re-combining, the two laser beams have acquired a phase difference

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$(2\pi/\lambda)|\Delta L - \Delta L'| = (2\pi a/\lambda) A \cos(\omega t)$

The apparatus is a Michelson interferometer. This phase difference between the reflected beams gives an interference maxima when it equals an even multiple of π, and an interference minimum when it equals an odd-number of half-π's. By watching for these interference maxima and minima, one can measure changes in the relative lengths of the x and y distances due to the passage of a gravitational wave.

Join the fun and do some serious physics by registering yourself and your computer on the LIGO data analysis team! To register your computer to analyze LIGO data, log on to:
www.physics2005.org/events/einsteinathome/index.html

Bibliography:
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